1. Consider the parabola $y = x^2$ in $\mathbb{R}^2$.
   (a) Find a polynomial $f \in \mathbb{R}[x, y, z]$ so that the variety $V$ in $\mathbb{P}^2_{\mathbb{R}}$ defined by $f$ has $V \cap \mathbb{R}^2$ the above parabola. Here $\mathbb{R}^2 = \{(x : y : 1)\}$.
   (b) How many points does $V$ have in the line at infinity $\mathbb{P}^1_{\infty} = \{(x : y : 0)\}$?
   (c) Using a projective transformation, show that $V$ is in fact tangent to $\mathbb{P}^1_{\infty}$.
   (d) Find a projective transformation so that $p_A(V) \cap \mathbb{R}^2$ is a hyperbola.

2. Let $k$ be a field. A line in $\mathbb{P}^2_k$ is the variety corresponding to the equation $ax + by + cz = 0$, where $a, b, c \in k$ are not all zero.
   (a) Show that, up to change of coordinates, all lines are the same. That is, given two lines $L, L'$ there exists a matrix $A \in \text{GL}_3(k)$ so that the corresponding projective transformation $p_A: \mathbb{P}^2_k \rightarrow \mathbb{P}^2_k$ takes $L$ to $L'$.
   (b) Prove that any two distinct points $p_1, p_2 \in \mathbb{P}^2_k$ determine a unique line.
   (c) Prove that any two distinct lines intersect in exactly one point.
   **Hint:** What object in $k^3$ corresponds to a line in $\mathbb{P}^2_k$?

3. Let $k$ be a field, and consider $\mathbb{P} = \mathbb{P}^2_k$.
   (a) Let $p_1, p_2, p_3, p_4$ be points in $\mathbb{P}$ so that no three are colinear, i.e. no three lie on a common line. Show there is a projective change of coordinates so that the $p_i$ become $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)$.
   (b) Find all conics passing through the five points
   $$(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1), (a : b : c)$$
   (c) Suppose $p_1, \ldots, p_5$ are points in $\mathbb{P}$ with no four colinear. Use (a-b) to show there is at most one conic containing all 5 points.

   **Note:** This is one of many illustrations of the power of changing coordinates.

4. Consider the plane curve $X = \text{V}(x^3y + y^3z + z^3x)$ in $\mathbb{P}^2_{\mathbb{C}}$.
   (a) Find $X \cap L_{\infty}$, where $L_{\infty}$ is the line at infinity, i.e. $\text{V}(z)$.
   (b) Prove that $X$ is smooth, being sure to include those points found in (a).
   **Note:** Any smooth curve in $\mathbb{P}^2_{\mathbb{C}}$ is automatically irreducible, and has genus $\binom{d-1}{2}$, where $d$ is the degree of the defining polynomial. Hence, as topological space, $X$ is as shown below.
(c) $X$ is very symmetric. Find a group of projective transformations of order 21 that leaves $X$ invariant. In fact, the full group of such projective automorphisms has order 168 and is the simple group $\text{PSL}_2 \mathbb{F}_7$. In fact, this is the most symmetries that a genus 3 curve can have...

5. In this problem, you'll explore elliptic curves in $\mathbb{P}^2_{\mathbb{R}}$. In addition to the points in $\mathbb{R}^2$ given by a standard equation $y^2 = x(x^2 + ax + b)$, there is an additional point at infinity which is the identity element in the group law. Note: Elliptic curves are always taken to be smooth, as otherwise the group law gets confusing.

One thing that wasn't mentioned in class is how to add a point $p$ to itself. In this case, one takes the tangent line at $p$ as shown:

(a) Consider the curve $E$ given by $y^2 = x^3 + 4x$. Show that $(2, 4)$ has order 4.

(b) Now consider an arbitrary elliptic curve $E$. Explain why any point in $E$ of the form $(x, 0)$ has order 2 in $E$.

(c) Find the subgroup of $E$ consisting of all points of order 2 (plus the identity element), and identify it as a group. Note: there are two cases here, depending on the specific curve $E$. 