Math 418: HW 8 due Wednesday, April 13, 2022.

Webpage: http://dunfield.info/418

Office hours: Monday and Tuesday from 1:30–2:30pm; other times possible by appointment.

1. Fix a prime p. Show that the following subgroup of $\mathrm{GL}_2\mathbb{F}_p$ is solvable:

$$B = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \;\middle|\; x, y \in \mathbb{F}_p^\times, z \in \mathbb{F}_p \right\}$$

   Here, the group operation is just matrix multiplication.

2. (a) Prove directly from the definition that S_4 is solvable.

   (b) Prove that A_5 is simple using the following outline.

      (i) Show A_5 has 5 distinct conjugacy classes of elements, and count the number of elements in each class.

      (ii) For any normal subgroup $H \triangleleft G$ show that H is a union of conjugacy classes of G.

      (iii) If $N \triangleleft A_5$ use that $|N|$ divides $|A_5|$ and parts (i) and (ii) to show that $N = \{1\}$ or A_5.

      Alternatively, give a geometric proof using the fact that A_5 is the group of Euclidean isometries of a regular dodecahedron.

      Remark: A_5 is the smallest of all the simple groups. In fact, every group of order less than 60 is solvable.

   (c) Use (b) to show that S_n is not solvable for $n \geq 5$.

3. Let L be the Galois closure of a finite extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. If p is a prime dividing the order of $\mathrm{Gal}(L/\mathbb{Q})$, show that there is a subfield F of L with $[L : F] = p$ and $L = F(\alpha)$.

   Hint: You'll need to use Theorem 18 from Section 4.5: if p is a prime dividing the order of a finite group G, then G has an element of order p.

4. Let $F \subset \mathbb{R}$ be a field. Let a be an element of F which has a real n^{th} root $\alpha = \sqrt[n]{a}$, and set $K = F(\alpha)$. Prove that if L is any Galois extension of F contained in K then $[L : F] \leq 2$.

5. For a field k, here are some basic problems for varieties in k^2, where we take the coordinates to be (x, y). Except for part (b), *assume that k is algebraically closed.*

   (a) Let V be the x-axis, i.e. $V = \mathbf{V}(y)$. Prove that V is irreducible. Hint: Show a prime ideal is radical.

   (b) Give an example of a field k, necessarily not algebraically closed, for which the x-axis is *reducible.*

   (c) Prove that $V = \mathbf{V}(x - y)$ is irreducible.

   (d) Prove that $S = \{(a, a) \in k^2 \mid a \neq 1\}$ is *not* an algebraic variety if $k = \mathbb{C}$.

   (e) What is the decomposition of $V = \mathbf{V}(x^2 - y^2)$ into irreducibles? **Warning:** The answer depends on k!