Lecture 38: Orthogonal and unitary operators (§6.5)

Convention: Today, $V$ will always be a finite-dim'l inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

Story so far: $T$ linear op. on $V$.

Normal: $T \circ T^* = T^* \circ T$

Self-Adjoint: $T^* = T$

Isometry: $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$.

(aka orthogonal/unitary) [Recall examples...]

Thm: For a linear op $T$ on $V$, the following are equivalent:

a) $T$ is an isometry.

b) $\|T(x)\| = \|x\|$ for all $x \in V$.

c) $T \circ T^* = T^* \circ T = I_V$ ($\Rightarrow T$ normal)

d) For every orthonormal basis $\beta$ of $V$, the image $T(\beta)$ is also an orthonormal basis

e) For some orthonormal basis $\beta$ of $V$, $T(\beta)$ is orthonormal.
Proof: Learned (a) \(\iff\) (b) last time, and (d) \(\implies\) (e) is clear.

(a) \(\implies\) (d): Suppose \(\beta = \{u_1, \ldots, u_n\}\) and set \(w_i = T(u_i)\). As \(T\) is an isometry, have \(\langle w_i, w_j \rangle = \langle u_i, u_j \rangle\) and so \(\gamma = \{w_1, \ldots, w_n\}\) is also orthonormal. Moreover, \(\gamma\) is a basis since \(\#\gamma = \#\beta = \dim V\).

(c) \(\implies\) (b): \(\|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle x, T^* T(x) \rangle = \langle x, x \rangle = \|x\|^2\) for all \(x \in V\).

(e) \(\implies\) (c): Suppose \(\beta = \{u_1, \ldots, u_n\}\) is an orthonormal basis such that \(\gamma = \{w_1, \ldots, w_n\}\) with \(w_i = T(u_i)\) is also orthonormal. It suffices to show \(T^* \circ T = I_V\) as then \(T^* = T^{-1}\) and hence \(T \circ T^* = I_V\) as well. Now set \(v_i = T^* T(u_i)\).

Then \(\langle v_i, u_j \rangle = \langle T^* (T(u_i)), u_j \rangle = \langle T(u_i), T(u_j) \rangle = \langle w_i, w_j \rangle = \begin{cases} 1 & \text{if } i = j \smallskip \\ 0 & \text{otherwise} \end{cases}\)

Thus have \(v_i = u_i\) for all \(i\) and so \(T^* \circ T = I_V\). \(\square\)
If $T$ is an isometry of $V$ and $\beta$ an orthonormal basis for $V$, then by (c) we have

$$I_n = \begin{bmatrix} I_V \end{bmatrix}_\beta = \begin{bmatrix} T^* T \end{bmatrix}_\beta = \begin{bmatrix} T^* \end{bmatrix}_\beta \circ \begin{bmatrix} T \end{bmatrix}_\beta$$

$$= \left(\begin{bmatrix} T \end{bmatrix}_\beta\right)^* \begin{bmatrix} T \end{bmatrix}_\beta$$

Setting $A = \begin{bmatrix} T \end{bmatrix}_\beta$, have $A^* A = I_n$ and $A^{-1} = A^*$.

**Def:** A square matrix is **unitary** when $A^* A = I$.

It is **orthogonal** when $A^T A = I$.

So the matrix of an isometry with respect to an orthonormal basis is always unitary, and when $\mathbb{F} = \mathbb{R}$ it is also orthogonal.

**Thm:** Suppose $A \in M_{n \times n}(\mathbb{R})$ is orthogonal. Then $L_A$ is an isometry of $(\mathbb{R}^n, \text{dot})$.

**Note:** Analog is true for $A \in M_{n \times n}(\mathbb{C})$ that are unitary.
**Proof:** Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis for $\mathbb{R}^n$, which is orthonormal. Set $a_i = L_A(\mathbf{e}_i) = i^{th}$ column of $A$. Set $G = A^t A = \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$ and note $G_{ij} = a_i \cdot a_j$. Since $G = I$, this means $\{a_1, \ldots, a_n\}$ is orthonormal and hence $L_A$ is an isometry by (e).

**Cor:** For $A \in \text{Mat}_{m \times n}(\mathbb{R})$, the following are equivalent:

i) $A$ is orthogonal

ii) $A^t = A^{-1}$

iii) The columns of $A$ are an orthonormal basis for $(\mathbb{R}^n, \cdot, \cdot)$

iv) The rows of $A$

v) $L_A$ is an isometry of $(\mathbb{R}^n, \cdot, \cdot)$.

**Proof:** Exercise.

**Restated Thm:** Suppose $A \in \text{Mat}_{n \times n}(\mathbb{R})$ is symmetric.

Then there is an orthogonal $Q$ with $Q^t A Q = Q^{-1} A Q$ diagonal.
Operators in Quantum Mechanics:

\[ \mathcal{H} = \text{Hilbert Space} = \text{complex inner product space of the system} \]

**Ex 1:** Two particles, each of which is either "spin up" \( \uparrow \) or "spin down" \( \downarrow \).

\[ \mathcal{H} = 4 \text{ dim'l inner product space over } \mathbb{C} \]

with orthonormal basis \( \{ e_{\uparrow \uparrow}, e_{\uparrow \downarrow}, e_{\downarrow \uparrow}, e_{\downarrow \downarrow} \} \)

pure states.

**Ex 2:** Single particle moving in one dimension

\[ \mathcal{H} = L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \text{ "reasonably nice" } \} \]

\[ \langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx. \]

At time \( t \), state of system is described by a unit vector \( \psi_t \in \mathcal{H} \).

**Ex 1:** \( \psi_0 = \frac{2}{\sqrt{6}} e_{\uparrow \uparrow} + \frac{i}{\sqrt{6}} e_{\uparrow \downarrow} - \frac{1}{\sqrt{6}} e_{\downarrow \uparrow} \) (superposition of pure states.

Here, if measure system will find it in state \( \uparrow \uparrow \) with probability \( \frac{2}{6} \) and in state \( \uparrow \downarrow \) with prob \( \frac{1}{6} \).
Observables (position, momenta, energy, ...) are self-adjoint operators $A$ on $\mathcal{H}$.

\[ \text{Ex 1: (1st particle is spin up) = projection onto } \text{span}\{e_{\uparrow\uparrow}, e_{\downarrow\downarrow}\} \]

Expected values of an observable $A$ is computed in terms of decomposition of $\psi_t$ as a linear combination of eigenvectors of $A$ (which is diagonalizable!)

The operator corresponding to the total energy of the system is the Hamiltonian $H$. The time evolution of the system is governed by Schrödinger's equation

\[ i\hbar \frac{\partial}{\partial t} \psi_t = H \psi_t \]

which implies $\psi_t = U_t \psi_0$ where $U_t$ is the unitary transformation $U_t = \exp(-i t H)$. Here, $I$ am implicitly using the matrix exponential:

\[ \exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \text{ for } X \in M_{n \times n}(\mathbb{C}). \]