Lecture 32: More on inner products (§ 6.1 and 6.2)

Last time: V a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An inner product is a $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{F}$ satisfying:

a) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

b) $\langle cx, y \rangle = c \langle x, y \rangle$

c) $\langle y, x \rangle = \overline{\langle x, y \rangle}$

d) $\langle x, x \rangle$ is in $\mathbb{R}_{\geq 0}$ for $x \neq 0$.

For all $x, y, z \in V$ and $c \in \mathbb{F}$.

From $\langle \cdot, \cdot \rangle$, we define $\|x\| = \sqrt{\langle x, x \rangle}$ for $x \in V$ and call it the norm/length of $x$.

Thm: Suppose $V$ is an inner product space. For all $x, y \in V$ and scalar $c$ one has:

a) $\|cx\| = |c| \|x\|$

b) $\|x\| \iff x = 0$

c) $|\langle x, y \rangle| \leq \|x\| \|y\|$ (Cauchy-Schwartz)
d) \[ \|x+y\| \leq \|x\| + \|y\| \ (\Delta - \text{inequality}) \]

Proof: Parts (a) and (b) are easy; will do (c) and (d) when the field of scalars is \( \mathbb{R} \).

c) If \( y = 0 \) get \( 0 \leq 0 \) as needed, so assume \( y \neq 0 \). Notice that if we scale \( y \) by \( c \in \mathbb{R} \), then both sides of (c) change by \( |c| \) as per part (a).

Replacing \( y \) with \( \frac{1}{\|y\|} y \) we can thus assume that \( \|y\| = 1 \). Now

\[
0 \leq \|x - \langle x, y \rangle y\|^2 = \langle x, x \rangle - 2\langle x, y \rangle^2 + \langle x, y \rangle^2 \langle y, y \rangle = \|x\|^2 - \langle x, y \rangle^2
\]

and so \( |\langle x, y \rangle| \leq \|x\| \) as needed.

d) \[
\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\
\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\
\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \ (\text{by (c)}) \\
= (\|x\| + \|y\|)^2
\]
Def: Suppose $V$ is an inner product space. Vectors $x, y \in V$ are orthogonal/perpendicular when $\langle x, y \rangle = 0$. A subset $S \subseteq V$ is orthogonal when each pair of distinct vectors is orthogonal. A vector $x \in V$ is unit when $\|x\| = 1$. Finally, a subset of $V$ is orthonormal when it is orthogonal and consists entirely of unit vectors.

Ex: $\mathbb{R}^2$, dot product

- $S = \{e_1, e_2\}$ is orthonormal
- $S = \{2e_1 + 2e_2, e_1 - e_2\}$ is orthogonal.
- $S = \{e_1, e_1 + e_2\}$ is not orthogonal. $S = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$ is orthonormal

Def: An orthonormal basis for an inner product space $V$ is a basis which is orthonormal.

Thm: An finite dim'l inner product space has an orthonormal basis.

[Will prove this next time... ]
Thm: Suppose \( S = \{v_1, v_2, \ldots, v_k\} \) is an orthonormal subset of an inner product space \( V \).

If \( y \in \text{span}(S) \), then \( y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i \).

Ex: \((\mathbb{R}^3, \text{dot product})\) \( S = \{e_1, e_2, e_3\} \)

\( y = (4, 3, -2) = 4e_1 + 3e_2 - 2e_3 \)

\( \langle y, e_i \rangle = 4 \quad \langle y, e_2 \rangle = 3 \quad \langle y, e_3 \rangle = -2 \).

Proof: As \( y \in \text{span}(S) \), there are scalars \( a_i \) with

\( y = \sum_{i=1}^{k} a_i v_i \).

Then

\[ \langle y, v_j \rangle = \sum_{i=1}^{k} \langle a_i v_i, v_j \rangle \]

\[ = \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle = a_j \quad \text{for all } 1 \leq j \leq k. \]

\[ = 0 \quad \text{when } i \neq j \]

\[ = \|v_j\|^2 = 1 \quad \text{when } i = j \]

Thus

\[ \sum_{i=1}^{k} \langle y, v_i \rangle v_i = \sum_{i=1}^{k} a_i v_i = y \quad \text{as needed} \]

Cor: Suppose \( S = \{v_1, \ldots, v_k\} \) is orthonormal with all \( v_i \neq 0 \). If \( y \in \text{span}(S) \) then

\[ y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i \]
Proof: Set $\overline{V}_i = \frac{V_i}{\|V_i\|}$ so that $S = \{\overline{V}_1, \ldots, \overline{V}_k\}$ is orthonormal. As $\text{span}(S) = \text{span}(S)$, given $y \in \text{span}(S)$ we have

$$y = \sum_{i=1}^{k} \langle y, \overline{V}_i \rangle \overline{V}_i = \sum_{i=1}^{k} \frac{\langle y, V_i \rangle}{\|V_i\|^2} V_i.$$ 

Cor: Suppose $S \subseteq V$ is orthogonal and $0 \notin S$. Then $S$ is linearly independent.

Pf: Since scaling vectors by non-zero amounts does not change dependence, it suffices to show that any orthonormal $V_1, \ldots, V_k$ are linearly independent. If $\sum_{i=1}^{k} a_i V_i = 0$, then as before we have $0 = \langle v_j, 0 \rangle = \langle v_j, \sum_{i=1}^{k} a_i V_i \rangle = a_j$

for each $j$. Thus $V_1, \ldots, V_k$ is linearly independent as needed.
Gram-Schmidt Process: Suppose \( \{w_1, w_2, \ldots, w_n\} \) is a linearly independent subset of \( V \). Set

\[
V_1 = \text{unit}(w_1) = \frac{w_1}{||w_1||}
\]

\[
V_2 = \text{unit}(w_2 - \langle w_2, v_1 \rangle v_1)
\]

\[
V_3 = \text{unit}(w_3 - (\langle w_3, v_1 \rangle v_1 + \langle w_3, v_2 \rangle v_2))
\]

and in general

\[
V_k = \text{unit}(w_k - \sum_{i=1}^{k-1} \langle w_k, v_i \rangle v_i)
\]

Underlying geometry for \( V_3 \):

\[
\langle w_3, v_1 \rangle v_1 + \langle w_3, v_2 \rangle v_2
\]