Lecture 31 Inner product spaces. (§6.1)

[Original motivation was the geometry/algebra of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. One thing that has been missing: the dot product. Let's fix that...]

Def: Suppose V is a vector space over $\mathbb{R}$. An *inner product* is a function from pairs of vectors to $\mathbb{R}$, denoted $\langle x, y \rangle$, such that for all $x, y, z \in V$ and $c \in \mathbb{R}$ one has:

a) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

b) $\langle cx, y \rangle = c \langle x, y \rangle$

c) $\langle x, y \rangle = \langle y, x \rangle$

d) $\langle x, x \rangle > 0$ if $x \neq 0$.

Ex: $V = \mathbb{R}^n$, $\langle x, y \rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$

Ex: $V = \mathbb{R}^2$ $\quad \langle x, y \rangle = 3x_1 y_1 + 2x_2 y_2$

Ex: $V = \mathbb{R}^2$ $\quad \langle x, y \rangle = 2x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2$

<u>Non Ex</u>: $V = \mathbb{R}^2$ with any of

$$\langle x, y \rangle = x_1^2 y_1 + x_2 y_2 \qquad \langle x, y \rangle = x_1 y_1 + x_1 y_2 + x_2 y_2$$

$$\langle x, y \rangle = x_1 y_1 - x_2 y_2$$

<u>Ex</u>: $V = \mathcal{C}([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid f \text{ continuous} \}$

$$\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$$

For (d), have $\langle f, f \rangle = \int_0^1 (f(t))^2 dt$, and so if this $= 0$ we must have $f^2 = 0$ (as $f^2 \geq 0$) and so $f = 0$.

<u>Def</u>: Suppose V is a vector space over $\mathbb{C}$. An <u>inner product</u> on V is a function from pairs of vectors to $\mathbb{C}$ satisfying the same rules with one change:

c') $\overline{\langle x, y \rangle} = \langle y, x \rangle$

Here "$-$" denotes complex conjugation: $\overline{a+bi} = a - bi$

<u>Non Ex</u>: $V = \mathbb{C}^2 \quad \langle x, y\rangle = x_1 y_1 + x_2 y_2$

<u>Problem</u>: $\langle (i,0), (i,0)\rangle = -1$

<u>Ex</u>: $V = \mathbb{C}^2 \quad \langle x, y\rangle = x_1 \overline{y_1} + x_2 \overline{y_2}$

Now $\langle (i,0), (i,0)\rangle = i \cdot \overline{i} + 0 \cdot \overline{0} = +1 > 0.$

<u>Def</u>: The <u>conjugate transpose</u> or <u>adjoint</u> of $A \in M_{n\times n}(\mathbb{C})$ is $A^* = \overline{(A^t)}$. That is $A^*_{ij} = \overline{A_{ji}}$

<u>Ex</u>: $A = \begin{pmatrix} 2+i & 3 \\ -i & 4 \end{pmatrix} \qquad A^* = \begin{pmatrix} 2-i & i \\ 3 & 4 \end{pmatrix}$

For $A \in M_{n\times n}(\mathbb{R})$, A^* is just A^t. As with transpose, have $(A+B)^* = A^* + B^*$ and $(AB)^* = B^* A^*$.

The <u>Frobenius inner product</u> on $M_{n\times n}(\mathbb{R})$ or $M_{n\times n}(\mathbb{C})$ is defined by $\langle A, B\rangle = tr(B^* A)$.

This satisfies (a) and (b) by linearity of tr, satisfies (c) because $tr(C^*) = \overline{tr(C)}$ and $(B^*A)^* = A^*B$.

Finally, note that the diagonal entries of A^*A are the standard inner products of the columns of A.

In particular, $(A^*A)_{ii} \geq 0$ and is $= 0$ only when the i^{th} column of A is 0. So $\langle A, A\rangle = 0$ implies $A = 0$, completing the proof of (d).

<u>Basic Properties</u>: V an <u>inner product space</u>.

A pair $(V, \langle\ ,\ \rangle)$

For $x, y, z \in V$ and any scalar c:

$$\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$$

$$\langle x, cy\rangle = \bar{c}\langle x, y\rangle$$

$$\langle x, 0\rangle = \langle 0, x\rangle = 0$$

$$\langle x, x\rangle = 0 \iff x = 0$$

<u>Def</u>: Let V be an inner product space. The <u>norm</u> or <u>length</u> of $x \in V$ is $\|x\| = \sqrt{\langle x, x\rangle} \in \mathbb{R}_{\geq 0}$.

<u>Ex</u>: ($\mathbb{R}^n$, dot product) Then $\|x\| = \sqrt{\langle x, x\rangle} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$.

<u>Thm</u>: Suppose V is an inner product space. For all $x, y \in V$ and scalars c one has

a) $\|cx\| = |c|\,\|x\|$

b) $\|x\| = 0 \iff x = 0$

c) $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz)

d) $\|x+y\| \leq \|x\| + \|y\|$ (Δ- inequality)

<u>Proof</u>: Parts (a) and (b) are easy; will do (c) and (d) in the case where the scalars are $\mathbb{R}$.

c) If $y=0$, we get $0 \leq 0$ as need; so assume $y \neq 0$. Also if scale x or y by $c \in \mathbb{R}$, then both sides change by $|c|$ as per part (a). So may assume that $\|y\|=1$. Now

$$0 \leq \| x - \langle x,y\rangle y \|^2 = \langle x,x\rangle - 2\langle x,y\rangle^2 + \langle x,y\rangle^2 \langle y,y\rangle = \|x\|^2 - \langle x,y\rangle^2$$

$$\| x - \langle x,y\rangle y \|^2 = \langle x - \langle x,y\rangle y,\, x - \langle x,y\rangle y\rangle$$

and so $|\langle x,y\rangle|^2 \leq \|x\|$ as needed.

d) $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$

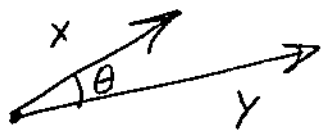
$$\leq \langle x,x\rangle + 2|\langle x,y\rangle| + \langle y,y\rangle$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

$\square$

Note: Motivation for (c) is that in $\mathbb{R}^3$ one has $x \cdot y = \|x\| \|y\| \cos\theta$



Def: Suppose V is an inner product space. Vectors $x, y \in V$ are orthogonal/perpendicular if $\langle x, y \rangle = 0$. A subset $S \subseteq V$ is orthogonal if every pair of distinct vectors in it are orthogonal. A vector $x \in V$ is unit if $\|x\| = 1$. Finally, a subset $S \subseteq V$ which is orthogonal and consists of unit vectors is called orthonormal.

Ex: The standard basis $\{e_i\}$ is an orthonormal subset of $\mathbb{R}^n$ with $\langle\, ,\, \rangle =$ dot product.