Lecture 3: Subspaces (§1.3 of [FIS])

Previously on Math 416...

A vector space over $\mathbb{R}$ is a set $V$ with two operations (vector addition and scalar multiplication) satisfying:

1. Vector addition is commutative and associative.
2. There is a zero vector.
3. Additive inverses exist.
4. Scalar multiplication is associative.
5. $1v = v$
6. Scalar multiplication is associative.
7-8. Distributive properties.

Ex: $\mathbb{R}^n$, $\text{Mat}_{m \times n}(\mathbb{R})$, spaces of functions...

Back to $\mathbb{R}^3$: Other basic objects: lines and planes.

Today: Analog of such in a general vector space.
Suppose $W$ is a plane in $\mathbb{R}^3$ containing $O$, and $w_1$, $w_2$ are vectors in $W$. Then $w_1 + w_2$ is also in $W$. So is $aw$, for any $a$ in $\mathbb{R}$.

Note: Important that $W$ contains $O$ here as otherwise these props need not hold.

Def: Suppose $V$ is a vector space over $\mathbb{R}$. A subset $W$ of $V$ is a Subspace if

(a) $O$ is in $W$

(b) For all $w_1, w_2$ in $W$, the sum $w_1 + w_2$ is also in $W$.

(c) For all $c$ in $\mathbb{R}$ and $w$ in $W$, $cw$ is also in $W$.

[Can replace (a) with requirement that $W$ is nonempty.]
Ex: Some subspaces of $\mathbb{R}^3$:
1. $\mathbb{R}^3$
2. $\{0\}$
3. $\{(x,0,0) \text{ for } x \text{ in } \mathbb{R}\}$
4. $\{(x,-x,2x)\}$
5. $\{(x,y,0)\}$
6. $\{x+y+z=0\} = \{s(1,0,-1)+t(1,-1,0) \text{ for } s,t \text{ in } \mathbb{R}\}$

Ex: In any vector space $V$, the subsets $\{0\}$ and $V$ are subspaces.

Thm: Suppose $W$ is a subspace of a vector space $V$. Then $W$ is itself a vector space under the two operations inherited from $V$.

Proof: First by requirements 6 and 10 we do have two ops taking values in $W$. Of the 8 conditions, 1-2 and 5-8 are immediate from the fact that $V$ itself is a vector space. Moreover, 3 follows from subspace cond. 2.
Finally, for \( \circ \) given \( W \) in \( W \) we know there is a \( v \) in \( V \) such that \( v + w = 0 \).

**Issue:** Does \( v \) have to be in \( W \)?

Yes, since we can take \( W = (-1)v \) which is in \( W \) by \( \circ \).  \[ \text{Check: } v + (-1)v = 1v + (-1)v = (1-1)v = ov = 0 \]

↑ Think of last time.

So \( W \) with these ops satisfies \( \circ \circ \circ \) and so is a vector space.

\[ \text{Non-Ex}: \quad W = \{ (w_1, w_2) \text{ with } w_i \geq 0 \} \]

in \( \mathbb{R}^2 \) is not a subspace.

\[ \text{Satisfies } \circ \text{ and } \circ \text{ but not } \circ \circ \circ \].

In proof of thm, everything works except \( \circ \).

[Discuss difference with book's treatment of subspaces]
Ex: $A$ in $\text{Mat}_{n \times n}(\mathbb{R})$

$$
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{pmatrix} = (A_{i,j})
$$

Transpose: $A^t$ where $A^t_{ij} = A_{ji}$

$$
(1\ 2)^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}
$$

[Also works for non-square matrices.]

A matrix $A$ in $\text{Mat}_{nxn}(\mathbb{R})$ is symmetric if $A = A^t$.

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ but neither of the two examples above.

Thm: The subset of symmetric matrices in $\text{Mat}_{nxn}(\mathbb{R})$ is a subspace.
Proof: The 0 in \( \text{Mat}_{n \times n}(\mathbb{R}) \) is \((0 \cdots 0)\) which is symmetric so 0 holds.

For (b) and (c), first show that for all \( A, B \) in \( \text{Mat}_{n \times n}(\mathbb{R}) \) and \( a, b \) in \( \mathbb{R} \) one has

\[
(aA + bB)^t = a(A^t) + b(B^t).
\]

Now if \( A, B \) are sym, then

\[
(A + B)^t = A^t + B^t = A + B
\]

and so \( A + B \) is also sym, proving (b).

The argument for (c) is similar.

Next time: Linear combinations and linear equations.