Lecture 28: Proof of the diagonalization criteria.

Last time: \( \lambda \) eigenvalue of \( A \in M_{n \times n}(\mathbb{R}) \).

**Multiplicities:**

- **Algebraic:** \# of times \((t - \lambda)\) divides the char poly of \( A \).
- **Geometric:** \( \dim E_\lambda \)

**Thm:** \( A \in M_{n \times n}(\mathbb{R}) \) is diagonalizable if and only if

a) The char poly of \( A \) splits completely over \( \mathbb{R} \)

b) \((\text{alg. mult}) = (\text{geom mult})\) for all eigenvalues of \( A \).

**Lemma:** Suppose \( v_1, v_2, \ldots, v_k \in \mathbb{R}^n \) are eigenvectors of \( A \) corresponding to distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_k \). Then \( \{v_1, \ldots, v_k\} \) are linearly independent.

**Moral:** Can't create an eigenvector with eigenvalue \( \lambda \) from eigenvectors with other eigenvalues.
Ex: \[ A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \]

\[ v_1 = e_1, \quad v_2 = e_2 \]

\[ A(e_1 + e_2) = Ae_1 + Ae_2 = 2 \cdot e_1 + 3 \cdot e_2 \]

**Proof of Lemma:** Can assume \( \lambda_k \neq 0 \) for \( k > 1 \). Induction on \( k \).

**Base case:** As \( v_1 \) is an eigenvector, it is nonzero and so \( \{v_1\} \) is linearly independent.

**Inductive Step:** Assume \( \{v_1, v_2, \ldots, v_{k-1}\} \) is linearly independent. Will prove by contradiction, so assume \( \{v_1, \ldots, v_k\} \) is linearly dependent.

Then \( v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1} \) for some \( a_i \in \mathbb{R} \).

Now \[ Av_k = \lambda_k v_k = \sum_{i=1}^{k-1} (\lambda_k a_i) v_i \]

and \[ Av_k = \sum_{i=1}^{k-1} A(a_i v_i) = \sum_{i=1}^{k-1} (\lambda_i a_i) v_i \]
Now $A v_k \neq 0$ but we have two distinct ways of writing it as a linear combination of the linearly indep. set $\{v_1, \ldots, v_{k-1}\}$, which is impossible. So $\{v_1, \ldots, v_k\}$ is linearly independent, completing the induction. \(\square\)

Lemma: Suppose $\lambda_1, \ldots, \lambda_k$ are eigenvalues for $A$. If $\beta_i \subseteq E_{\lambda_i}$ is lin. indep., then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is linearly independent.

Note: $\beta_i \cap \beta_j = \emptyset$ for $i \neq j$ as if $v \neq 0$ then $Av = \lambda_i v$ and $Av = \lambda_j v \Rightarrow \lambda_i = \lambda_j$.

Proof: Suppose $\beta_i = \{v_1^i, v_2^i, \ldots, v_{d_i}^i\}$ and there are scalars $a_j^i$ such that

$$
\sum_{i=1}^{k} \sum_{j=1}^{d_i} a_j^i v_j^i = 0
$$

Each $w_i \in E_{\lambda_i}$ and is either an eigenvector.
for $\lambda; n$ is 0. By earlier lemma, can't have a linear dependence among eigenvectors with different eigenvalues, so all $w_i = 0$. As each $\beta_i$ is linearly indep, conclude $a_{ij} = 0$ for all $i$ and $j$. So $\beta$ is linearly independent.

**Proof of Thm:** $(\Leftarrow)$ By (a) have

\[
\text{char poly of } A = \pm (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}
\]
for distinct $\lambda_i \in \mathbb{R}$. Notice $\sum_{i=1}^{k} m_i = \text{deg (char poly)} = n$. Let $\beta_i$ be a basis for $E_{\lambda_i}$. By (b), know $\#\beta_i = \text{dim } E_{\lambda_i} = m_i$. Set $\beta = \beta_1 \cup \cdots \cup \beta_k$.

Now $\#\beta = \sum \#\beta_i = n$ and $\beta$ is lin. indep by the lemma. So $\beta$ is a basis of $\mathbb{R}^n$, consisting of eigenvectors for $A$, and so $A$ is diagonalizable.
(⇒) Suppose $A$ is diagonalizable. Showed last time that (a) follows, so let $\lambda_i, m_i$ be as before. Set $d_i = \dim E_{\lambda_i}$, which we know satisfies $d_i \leq m_i$. Let $\beta$ be a basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$. Set $b_i = \# \text{ of } v \text{ in } \beta \text{ that are in } E_{\lambda_i}$. Know $b_i \leq d_i \leq m_i$ and so

$$n = \# \beta = \sum b_i \leq \sum d_i \leq \sum m_i = n$$

Thus we must have $b_i = d_i = m_i$ for all $i$, proving (b). Q.E.D.