Lecture 27: Diagonalization Criteria [§5.2]

<u>Thm</u>: $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if there is a basis of $\mathbb{R}^n$ consiting of eigenvectors for A.

<u>Def</u>: If λ is an eigenvalue for $A \in M_{n\times n}(\mathbb{R})$ then

$$E_\lambda = \{v \in \mathbb{R}^n \mid Av = \lambda v\} = \mathcal{N}(A - \lambda I_n)$$

is called the <u>eigenspace</u> of A corresponding to λ.

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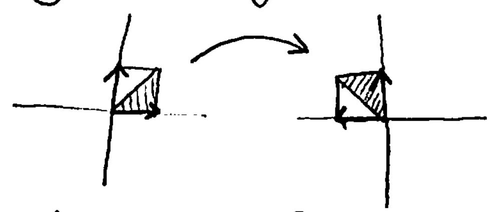
<u>Today's Moral</u>: There are two things that can prevent diagonalization

① <u>Too few eigenvalues</u>: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\text{Char poly} = \det(A - tI_2) = \det\begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix}$$
$$= t^2 + 1.$$

This has no roots in $\mathbb{R}$, so A has no eigenvalues/eigenvectors at all!

Note: Geometrically, $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation by $\pi/2$, so clearly no eigenvectors.



Potential fix: Enlarge field to $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$ where $i^2 = -1$. See HW for more.

2 Too few eigenvectors: $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then char poly $= (t-1)^2$ and so all eigenvectors live in:

$$E_1 = \mathcal{N}\left(B - I_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \{(t, 0) \mid t \in \mathbb{R}\}$$

So any eigenvector is a scalar mult of any other $\Longrightarrow$ Can't have a basis of eigenvectors, so not diagonalizable.

Def: A polynomial $f(t)$ with coefficents in a field $\mathbb{F}$ splits (or splits completely) over $\mathbb{F}$ if $f(t) = c(t-a_1)(t-a_2)\cdots(t-a_d)$

where $c, a_1, \ldots, a_d \in \mathbb{F}$ and $d = \deg(f)$.

Fundamental Theorem of Algebra: Every polynomial with coeffs in $\mathbb{C}$ splits completely over $\mathbb{C}$.

Thm: If $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable, then its char. poly splits completely over $\mathbb{R}$.

Proof: As A is similar to a diagonal matrix, there is a $Q \in M_{n\times n}(\mathbb{R})$ with $Q^{-1}AQ = D$
$= \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$. Then $\det(A - tI_n) =$

$\det(QDQ^{-1} - Q(tI_n)Q^{-1}) =$

$\det(Q(D-tI)Q^{-1}) = \det \begin{pmatrix} \lambda_1 - t & & 0 \\ 0 & \lambda_2 - t & \\ & & \ddots \end{pmatrix}$

$$= (\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t)$$

and so the char poly of A splits completely. ▨

[This gives a necessary condition for A to be diagonalizable. Is it also sufficient? No!]

<u>Def</u>: Suppose λ is an eigenvalue of A. The <u>algebraic multiplicity</u> of λ is the number of times that $(t - \lambda)$ divides the char poly of A. The <u>geometric multiplicity</u> of λ is $\dim(E_\lambda)$.

<u>Thm</u>: For each λ, (geom mult) $\leq$ (alg mult)

<u>Thm</u>: A matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if

a) char poly of A splits completely.

b) For every eigenvalue λ, (geom mult) = (alg. mult.)

Cor: If the char poly of $A \in M_{n\times n}(\mathbb{R})$ has n distinct roots in $\mathbb{R}$ then A is diagonalizable.

Pf of Cor: If the char poly has n distinct roots and degree n, then it splits completely. Moreover every eigenvalue has alg. mult. 1. As the geom mult of any λ is ≥ 1 we thus have that all alg and geom. mult. agree and so the theorem applies. ▨

Pf of Thm about multiplicities: Let $\{v_1, \dots, v_k\}$ be a basis for E_λ. Enlarge this to a basis $\beta = \{v_1, v_2, \dots, v_n\}$ for $\mathbb{R}^n$. Then

$$[L_A]_\beta = \left(\begin{array}{c|c} \underbrace{\begin{matrix} \lambda & 0 & \\ 0 & \lambda & \\ & & \lambda \end{matrix}}_{k} & * \\ \hline O & * \end{array}\right)\Big\}k \quad \text{as } L_A(v_i) = \lambda v_i \text{ for } 1 \leq i \leq k.$$

Setting $Q = [I_{\mathbb{R}^n}]_\beta^{std} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$, have

$[L_A]_\beta = Q^{-1}AQ$. As before, we see similar matrices have the same char. poly, and so

$$\text{Char poly } A = \det([L_A]_\beta - tI_n)$$
$$= \det\left(\begin{array}{cc|c} \lambda - t & 0 & \\ 0 & \lambda - t & * \\ \hline \multicolumn{2}{c|}{0} & ? \end{array}\right)$$
$$= (\lambda - t)^k \det(?)$$

Thus (alg mult λ) $\geq k$ as required. $\blacksquare$