Lecture 24: Determinant wrap-up.

Last lecture:

Thm: $A \in M_{n \times n}(\mathbb{R})$ is invertible iff it is the product of elementary matrices.

Thm: $A, B \in M_{n \times n}(\mathbb{R})$. Then $\det(AB) = \det(A) \det(B)$.

Thm: $A \in M_{n \times n}(\mathbb{R})$. Then $\det(A) \neq 0$ if and only if $A$ is invertible.

Proof: If $A$ is not invertible, saw last time that $\det(A) = 0$. So assume $A$ is invertible. Then $\det(A) \cdot \det(A^{-1}) = \det(AA^{-1}) = \det(\mathbf{I}_n) = 1$.

So $\det(A) \neq 0$.

Thm: For $A \in M_{n \times n}(\mathbb{R})$, have $\det(A^T) = \det(A)$

Proof: By practice exam know $\dim(\text{RowSp}(A^T)) = \dim(\text{ColSp}(A))$ or $\text{rank}(A) = \dim(\text{RowSp}(A)) = \dim(\text{ColSp}(A))$.

Thm: $\text{rank}(A) = \text{rank}(A^T)$. 


If $A, A^t$ have rank < $n$, have $\det(A) = 0$ and $\det(A^t) = 0$ which match. If instead they are both invertible, by theorem $A$ is the product of elementary matrices

$$A = E_1 E_2 \cdots E_k \cdots E_l$$

Now one can check that any elementary $E_k$ satisfies $\det(E_k^t) = \det(E_k)$. Thus

$$\det(A^t) = \det(E_l^t E_{l-1}^t \cdots E_2^t E_1^t)$$

$$= TT \det(E_k^t) = TT \det(E_k) = \det(A)$$

as required.

[Two weeks ago, started with a list of 4 properties of $\det$, the only one we're still missing is...]

**Thm:** Suppose $A \in M_{n \times n}(\mathbb{R})$. Then $\det(A)$ measures, with sign, how $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ changes volumes of objects. Specifically, if $S \subseteq \mathbb{R}^n$ is closed and bounded, then

$$\text{Volume}(L_A(S)) = |\det(A)| \cdot \text{Volume}(S)$$
Ex: $S = \text{unit (hyper)cube in } \mathbb{R}^n = \{(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_i \leq 1\}$

Then

$L_A(S) = \{\sum_{i=1}^n t_i a_i \mid 0 \leq t_i \leq 1\}$

where $a_i$ is the $i^{th}$ column of $A$.

Reason: $(t_1, \ldots, t_n) = \sum_{i=1}^n t_i e_i$

and $L_A(e_i) = i^{th}$ col of $A$.

By definition, something like $S$ is called an $n$-dimensional parallelepiped.

Cor: The volume of the parallelepiped determined by vectors $a_1, \ldots, a_n$ in $\mathbb{R}^n$ is $|\det(a_1, \ldots, a_n)|$

$= |\det(-\frac{a_1}{a_2} =) |$.

Note: For $n=3$, this was the triple product $a_1 \cdot (a_2 \times a_3)$. 
Sketch of ideas behind theorem.

- Any $L_A$ distorts volume “uniformly.”

- Define $J(A)$ to be in $\mathbb{R}_{\geq 0}$ so that
  \[ \text{Vol}(L_A(S)) = J(A) \cdot \text{Vol}(S) \]
  for all closed and bounded $S$.

- Study composition

So

\[ J(AB) = \frac{1}{2} J(A) J(B) \]

- Show $J(E) = |\det(E)|$ for elementary matrices.
Combining with result that every invertible matrix is a product of elementary ones gives the theorem.

What does $E$ do?

1. $I_n \xrightarrow{R_r \leftrightarrow R_s} E$: EX: $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is reflection in $y = x$.

In general, $E$ is reflection in the hyperplane $x_r = x_s$. This preserves volumes, meshing with $\det(E) = -1$.

2. $I_n \xrightarrow{cR_r} E$: just stretches by $c$ in $e_r$ direction, 

$$
\begin{pmatrix}
1 & 0 \\
0 & c
\end{pmatrix}
$$

changing vol by $|c| = |\det(E)|$.

3. $I_n \xrightarrow{cR_s + R_r} E$: EX: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\det = 1$.

Doesn't change volume by Cavalieri's principle.

Cross-sectional area doesn't change

$\Rightarrow$ Volume doesn't change.

Basis for usual multivariable integration and works in all dimensions.

The other way to prove the theorem is change of coordinates...