Lecture 15: Composition of linear transformations and matrix multiplication [§2.3 of FIS]

Last time: $T: V \to W$ linear

$\beta = \{v_1, \dots, v_n\}$ basis for V

$\gamma = \{w_1, \dots, w_m\}$ basis for W

$$[T]_\beta^\gamma = \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix} \in \mathrm{Mat}_{m \times n}(\mathbb{R})$$

Notes:

- For $T: V \to V$, use $[T]_\beta$ for $[T]_\beta^\beta$.
- For vector spaces V and W, set

$$\mathcal{L}(V, W) = \{T: V \to W \mid T \text{ linear}\}$$

and $\mathcal{L}(V) = \mathcal{L}(V, V)$.

Fun fact: $\mathcal{L}(V, W)$ is itself a vector space, specifically a subspace of $\mathcal{F}(V, W)$.

[Will come back to this, but for now see text for details.]

Consider two linear transformations

$$X \xrightarrow{S} Y \xrightarrow{T} Z$$

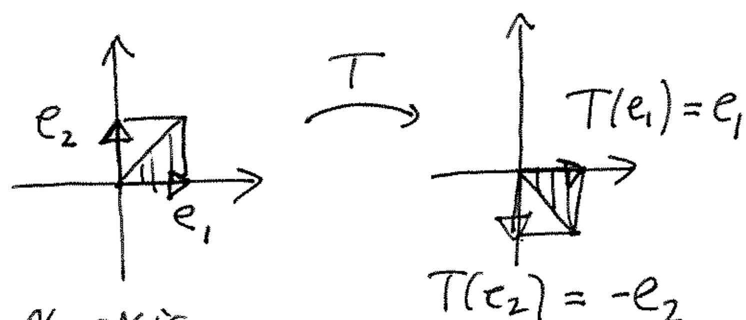
<u>Thm</u>: Their composition $T \circ S : X \to Z$, defined by $(T\circ S)(x) = T(S(x))$, is also linear.

<u>Pf</u>: Suppose $x_1, x_2 \in X$ and $a \in \mathbb{R}$. Then

$$\begin{aligned}(T\circ S)(ax_1 + x_2) &= T(S(ax_1 + x_2)) \\ &= T(aS(x_1) + S(x_2)) = aT(S(x_1)) + T(S(x_2)) \\ &= a(T\circ S)(x_1) + (T\circ S)(x_2) \quad \square\end{aligned}$$

<u>Ex</u>: $S, T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x,y) = (x, -y)$$

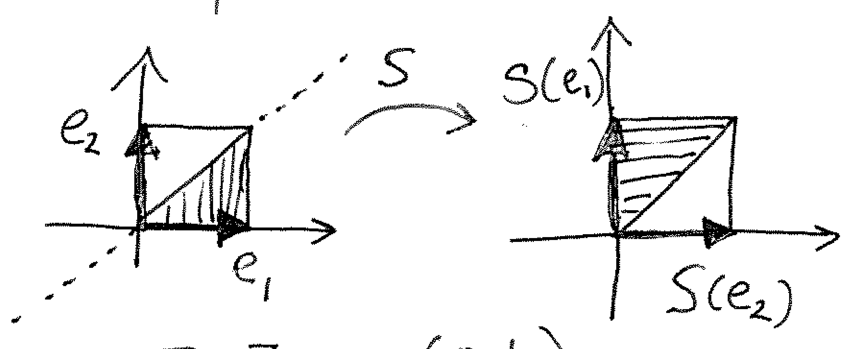


which is reflection across the x-axis.

For $\beta = \{e_1, e_2\}$ have $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

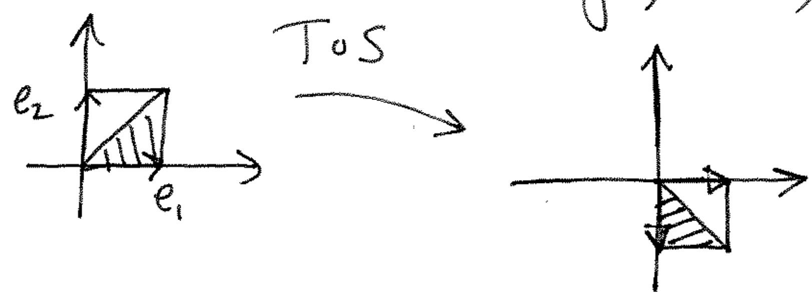
$$S(x,y) = (y,x)$$

which is reflection in $y = x$.



$$[S]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(T \circ S)(x,y) = T(S(x,y)) = T(y,x)$$
$$= (y, -x)$$



$$(T \circ S)(e_1) = -e_2$$
$$(T \circ S)(e_2) = e_1$$

So $T \circ S$ is rotation by $\pi/2$ clockwise and

$$[T \circ S]_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\alpha = \{x_1, \dots, x_n\}$ basis for X

$\beta = \{y_1, \dots, y_m\}$ basis for Y

$\gamma = \{z_1, \dots, z_p\}$ basis for Z

<u>Q</u>: How is $[T \circ S]_\alpha^\gamma$ related to $[S]_\alpha^\beta$ and $[T]_\beta^\gamma$?

[Answer involves matrix multiplication...]

<u>Def</u>: Suppose $A \in \mathrm{Mat}_{p \times m}(\mathbb{R})$ and $B \in M_{m \times n}(\mathbb{R})$. Their product $AB \in \mathrm{Mat}_{p \times n}(\mathbb{R})$ is defined by

$$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

Ex:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 8 & 7 \end{pmatrix}$$

$$\underset{2 \times 3}{} \qquad \underset{3 \times 2}{} \qquad \underset{2 \times 2}{}$$

Note: Viewing a column vector as an $m \times 1$ matrix, the above definition agrees with the product of (matrix)(vec) from last time. Moreover

$$AB = A \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{pmatrix} \qquad ✪$$

Thm: $\underset{p \times n}{[T \circ S]_\alpha^\gamma} = \underset{p \times m}{[T]_\beta^\gamma} \, \underset{m \times n}{[S]_\alpha^\beta}$

Pf: Recall that for $y \in Y$ we have

$$[T(y)]_\gamma = [T]_\beta^\gamma [y]_\beta$$

Now

$$[S]_\alpha^\beta = \begin{pmatrix} | & & | \\ [S(x_1)]_\beta & \cdots & [S(x_n)]_\beta \\ | & & | \end{pmatrix}$$

Using ✪ above, we have

$$[T]_\beta^\gamma [S]_\alpha^\beta = \underbrace{\begin{pmatrix} | \\ [T]_\beta^\gamma [S(x_i)]_\beta \\ | \end{pmatrix}}_{i=1,2,\ldots,n}$$

$$= \underbrace{\begin{pmatrix} | \\ [T(S(x_i))]_\gamma \\ | \end{pmatrix}}_{i=1,2,\ldots,n} = \underbrace{\begin{pmatrix} | \\ [(T\circ S)(x_i)]_\gamma \\ | \end{pmatrix}}_{i=1,2,\ldots,n}$$

$$= [T\circ S]_\alpha^\gamma$$ ▨

<u>Back to ex:</u> $T, S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad \beta = \{e_1, e_2\}$

$T =$ reflection in x-axis $\qquad S =$ reflection in $y = x$

Then
$$[T]_\beta [S]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which matches our calculation of

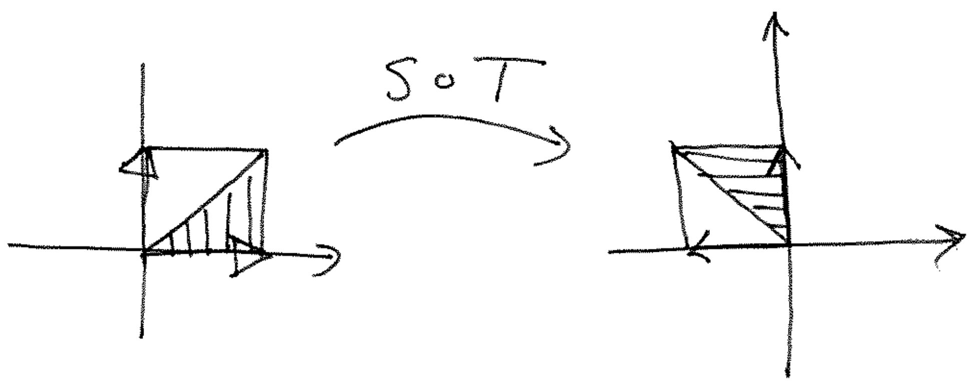
$$[T\circ S]_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

from before.

Important: Order of composition matters:

$$[S \circ T]_\beta = [S]_\beta [T]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which is not $[T \circ S]_\beta$!



As

$$(S \circ T)(e_1) = e_2$$

since $[S \circ T]_\beta [e_1]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $(S \circ T)(e_2) = -e_1$ since

$$[S \circ T]_\beta [e_2]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

So $S \circ T$ is rotation by $\pi/2$ counter-clockwise.

whereas $T \circ S$ is rotation by $\pi/2$ clockwise.