1. Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

(a) Find all the eigenvalues of A. Hint: They are all integers. **(4 points)**

Char poly $= \det(A - tI) = \begin{pmatrix} 3-t & -1 \\ 2 & -t \end{pmatrix} = t^2 - 3t + 2$

$= (t-1)(t-2)$

So eigenvalues are 1 and 2.

(b) For each eigenvalue find the corresponding eigenspace. **(3 points)**

$E_1 = \mathcal{N}(A - 1\cdot I) = \mathcal{N}\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} = \mathcal{N}\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$

$= \{(t, 2t) \mid t \in \mathbb{R}\} = \text{span}(\{v_1 = (1,2)\})$

$E_2 = \mathcal{N}(A - 2I) = \mathcal{N}\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$= \{(t,t) \mid t \in \mathbb{R}\} = \text{span}(\{v_2 = (1,1)\})$

<u>Quick checks</u>: $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c) Diagonalize A, giving both a diagonal matrix D and an invertible matrix Q such that $D = Q^{-1}AQ$. **(3 points)**

Take $Q = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Check:

$\underbrace{\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}_{Q^{-1}}\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$

2. Suppose $A \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ is a transition matrix associated to a Markov chain.

(a) If $1/2$ and $1/3$ are eigenvalues of A, prove that A is diagonalizable. **(5 points)**

As A is a transition matrix, it has 1 as an eigenvalue. Since the char poly $f(t)$ of A has degree 3 and each of $\{1, 1/2, 1/3\}$ is a root, we see it splits completely with all eigenvalues having algebraic mult 1. Since for each eigenvalue λ we have $1 \leq$ (geom mult) $\leq$ (alg mult) $= 1$, all the geom. multiplicities are 1. As the char poly of A splits completely and all algebraic and geometric multiplicities agree, A is diagonalizable.

(b) Give a transition matrix B where $1/2$ and $1/3$ are eigenvalues and justify your answer. **(2 points)**

Set $B = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix}$ which is transition since each column sums to 1. Its char poly is

$$\det(B - tI) = \begin{vmatrix} 1-t & 1/2 & 1/3 \\ 0 & 1/2 - t & 1/3 \\ 0 & 0 & 1/3 - t \end{vmatrix} = (1-t)(1/2-t)(1/3-t)$$

since $B - tI$ is upper triangular.

3. The *constant term* of a polynomial $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots a_1 t + a_0$ is the final coefficient a_0.

(a) For the characteristic polynomial of an $n \times n$ matrix A, prove that the constant term is $\det(A)$. Hint: for any polynomial $f(t)$ how does $f(0)$ relate to the constant term of $f(t)$? **(4 points)**

Solution: For any polynomial $f(t)$ as above, we have $f(0) = a_n \cdot 0^n + \cdots + a_1 \cdot 0 + a_0 = a_0$. Now let $g(t) = \det(A - tI_n)$ be the characteristic polynomial of A. Thus the contant term of g is $g(0) = \det(A - 0I_n) = \det(A)$ as needed.

(b) For the characteristic polynomial of an $n \times n$ matrix A, what is the coefficient a_{n-1} of t^{n-1} in terms of the entries of A? **(2 points)**

Solution: I will show $a_{n-1} = (-1)^{n-1}\operatorname{tr}(A)$, where $\operatorname{tr}(A) = \sum_{i=1}^n A_{ii}$ is the sum of the diagonal entries, using induction. The case of 1×1 matrices is immediate, so assume the claim holds for n. Set $B = A - tI_n$, and compute $\det(B)$ by expanding on the first row:

$$\det(B) = (A_{11} - t)\det \widetilde{B}_{11} + (\text{stuff})$$

Here, each term in "stuff" is $\pm A_{1j} \det \widetilde{B}_{1j}$. When $j > 1$, $\widetilde{B}_{1j}$ contains only $n-2$ rows with t's in them (since we've dropped B_{11} and B_{jj}) and hence $\det \widetilde{B}_{1j}$ has degree at most $n-2$. Hence the t^{n-1} term of $\det(B)$ comes entirely from $(A_{11} - t)\det \widetilde{B}_{11}$. By induction and fact that the leading term of the characteristic polynomial of an $(n-1) \times (n-1)$ matrix is $(-1)^{n-1} t^{n-1}$, we have

$$\begin{aligned}
(A_{11} - t)\det \widetilde{B}_{11} &= (A_{11} - t)\left((-1)^{n-1} t^{n-1} + (-1)^{n-2}\operatorname{tr}(\widetilde{A}_{11}) t^{n-2} + (\text{lower-order terms})\right) \\
&= (-1)^n t^n + (-1)^{n-1}(A_{11} + \operatorname{tr}(\widetilde{A}_{11})) t^{n-1} + (\text{lower-order terms}) \\
&= (-1)^n t^n + (-1)^{n-1}\operatorname{tr}(A) t^{n-1} + (\text{lower-order terms})
\end{aligned}$$

as needed to complete the induction.

4. Let V be an inner product space. A linear operator T on V is a *contraction* when $\|T(x)\| < \|x\|$ for all nonzero $x \in V$. Prove that if λ is an eigenvalue for a contraction T then $|\lambda| < 1$. **(4 points)**

Let $x \in V$ be an eigenvector with eigenvalue λ.

Then $\|T(x)\| = \|\lambda x\| = |\lambda| \, \|x\|$, as x is an eigenvector cor. to λ. Since T is a contraction and $x \neq 0$ (by def of eigenvector) we also have $\|T(x)\| < \|x\|$. Combining, get $|\lambda| \, \|x\| < \|x\|$ and since $\|x\| \neq 0$ can divide through to get $|\lambda| < 1$.

5. Suppose $\beta = \{u, v\}$ is an orthonormal basis for an inner product space V over $\mathbb{R}$.

(a) If $x = au + bv$, prove that $\|x\|^2 = a^2 + b^2$. **(4 points)**

$$\|x\|^2 = \langle x, x\rangle = \langle au + bv, au + bv\rangle$$

$$= \langle au, au\rangle + \langle au, bv\rangle + \langle bv, au\rangle + \langle bv, bv\rangle$$

$$= a^2\langle u, u\rangle + 2ab\langle u, v\rangle + b^2\langle v, v\rangle$$

$$= a^2 + 0 + b^2$$ since β is orthonormal, means $\langle u, v\rangle = 0$ and $\|u\| = \|v\| = 1$.

(b) For any $x \in V$, prove that $\|x\|^2 = |\langle x, u\rangle|^2 + |\langle x, v\rangle|^2$. **(3 points)**

Since β is a basis for V, $x \in \text{span}(\beta)$ and so $x = \langle x, u\rangle u + \langle x, v\rangle v$ since β is orthonormal. Taking $a = \langle x, u\rangle$ and $b = \langle x, v\rangle$ in part (a), we get $\|x\|^2 = a^2 + b^2 = \langle x, u\rangle^2 + \langle x, v\rangle^2$ as needed.

6. Circle the best answer for each part; you do **not** need to provide any justification. **(1 point each)**

(a) If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$ — true | false

$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(AB)$

(b) The matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is a product of elementary matrices. — true | false

$\det = -1 \neq 0 \Rightarrow$ invertible $\Rightarrow$ prod. of elem.

(c) If A and B are similar matrices, then A^3 and B^3 are also similar. — true | false

If $B = Q^{-1}AQ$ then $B^3 = Q^{-1}A^3Q$

(d) If all eigenvalues of $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ are 1 then $A = I_n$. — true | false

e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(e) The vectors $\{(1,0), (i,0), (0,1), (0,i)\}$ are a basis for $\mathbb{C}^2$ regarded as a vector space over $\mathbb{R}$. — true | false

(f) Exactly one of the following limits exist. Circle it.

$$\lim_{n\to\infty}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 0.4 & 0.5 \\ 0.6 & 0.5 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 0.2 & 0.3 \\ 0.9 & 0.8 \end{pmatrix}^n$$

regular transition matrix.

9 am

6. Circle the best answer for each part; you do **not** need to provide any justification. **(1 point each)**

(a) If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$ — true / false

$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$

(b) The matrix $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a product of elementary matrices. — true / false

$\det = 0 \Rightarrow$ not invert $\Rightarrow$ not a prod. of elem.

(c) If all eigenvalues of $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ are 0 then $A = 0$. — true / false

e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(d) If A and B are similar matrices, then A^2 and B^2 are also similar. — true / false

If $B = Q^{-1}AQ$ then $B^2 = Q^{-1}A^2Q$.

(e) The vectors $\{(1,0), (i,0), (0,1), (0,i)\}$ are a basis for $\mathbb{C}^2$ regarded as a vector space over $\mathbb{C}$. — true / false

$\dim_{\mathbb{C}} \mathbb{C}^2 = 2$ but there are 4 vectors here.

(f) Exactly one of the following limits exist. Circle it.

$$\lim_{n\to\infty}\begin{pmatrix} 0.4 & 0.5 \\ 0.8 & 0.7 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{pmatrix}^n$$

regular transistion matrix.

10 am

Extra Credit. Let $V = P(\mathbb{R})$ be the vector space of polynomials in x of arbitrary degree, that is, the union of all $P_n(\mathbb{R})$. Define an inner product on V by $\langle f(x),\ g(x)\rangle = \int_{-1}^{1} f(x)g(x)dx$. Consider the linear operator T on V given by $T(f(x)) = xf(x)$. Prove or disprove: T is a contraction in the sense of Problem 4. You can continue your answer on the back of this page if needed. **(2 points)**

Solution. Let $f(x) \in V$ be nonzero. Now $x^2 \leq 1$ for $x \in [-1,1]$ with $x^2 < 1$ for $x \in (-1,1)$. Hence $x^2(f(x))^2 \leq (f(x))^2$ for $x \in [-1,1]$ and so:

$$||T(f(x))||^2 = ||xf(x)||^2 = \int_{-1}^{1} x^2(f(x))^2 dx \leq \int_{-1}^{1} (f(x))^2 dx = ||f(x)||^2$$

To improve this to $||T(f(x))|| < ||f(x)||$, it suffices to show

$$0 < \int_{-1}^{1} (1 - x^2)(f(x))^2 dx.$$

The integrand is non-negative and is strictly positive at any $x \in (-1,1)$ which is not a root of the polynomial f. Since a nonzero polynomial f has only finitely many roots, the integrand is strictly positive except at finitely many points, forcing the integral to be strictly positive as well.