1. Suppose A and B are invertible matrices in $M_{n\times n}(\mathbb{R})$. Prove that $\det(ABA^{-1}B^{-1}) = 1$. **(5 points)** Ex:29

For any invertible matrix C, we have $\det(C)\cdot\det(C^{-1}) = 1$ since $\det(C)\cdot\det(C^{-1}) = \det(C\cdot C^{-1}) = \det(I_n) = 1$. Now

$$\det(ABA^{-1}B^{-1}) = \det(A)\det(B)\det(A^{-1})\det(B^{-1})$$
$$= \underbrace{\det(A)\det(A^{-1})}_{=1}\underbrace{\det(B)\det(B^{-1})}_{=1}$$
$$= 1$$

as required.

2. Consider the parallelepiped P in $\mathbb{R}^3$ determined by the three vectors $v_1 = (2, 0, 3)$, $v_2 = (0, 1, 2)$, and $v_3 = (0, 1, 0)$. Compute the *unsigned* volume of P. **(3 points)**

Set $A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 2 & 0 \end{pmatrix}$. Then $\det(A)$

$= 2\cdot\det\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = 2\cdot(-2) = -4.$

Hence the <u>unsigned</u> volume of P is $|\det(A)| = 4$.

3. Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

(a) Compute the characteristic polynomial of A. **(2 points)**

$$f(t) = \det(A - tI) = \det\begin{pmatrix} 3-t & 1 \\ 1 & 3-t \end{pmatrix} = (3-t)^2 - 1$$
$$= t^2 - 6t + 9 - 1 = t^2 - 6t + 8$$

(b) Find all the eigenvalues of A. Hint: They are all integers. **(2 points)**

$f(t) = t^2 - 6t + 8 = (t-2)(t-4) \Longrightarrow$ The eigenvalues of A ($=$ roots of $f(t)$) are 2 and 4.

(c) For each eigenvalue compute the corresponding eigenspace. **(2 points)**

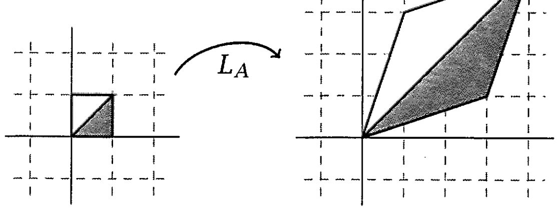
$$\lambda = 2: \quad E_2 = \mathcal{N}(A - 2 \cdot I) = \mathcal{N}\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \mathcal{N}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right)$$
$$= \text{span}(\{\underbrace{(1,-1)}_{v_2}\}) = \{(t,-t) \mid t \in \mathbb{R}\}.$$

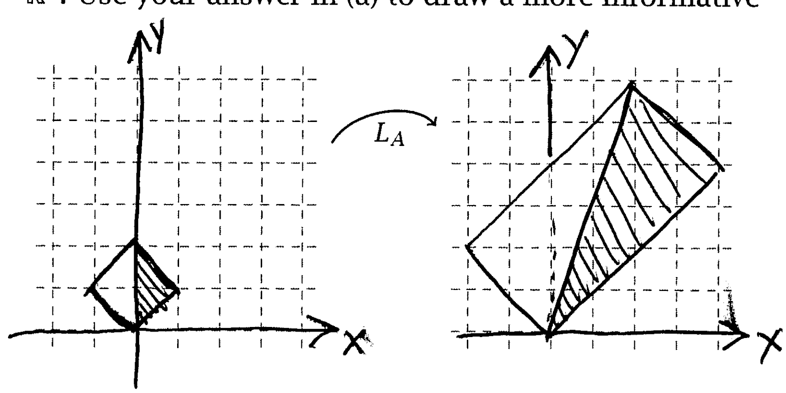
$$\lambda = 4: \quad E_4 = \mathcal{N}(A - 4 \cdot I) = \mathcal{N}\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right)$$
$$= \text{span}(\{\underbrace{(1,1)}_{v_4}\}) = \{(t,t) \mid t \in \mathbb{R}\}.$$

(d) Now diagonalize A, giving both a diagonal matrix D and an invertible matrix Q so that $D = Q^{-1}AQ$. **(3 points)**

Take $Q = \begin{pmatrix} | & | \\ v_2 & v_4 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ in which case $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

(e) At left is our usual visualization of $L_A\colon \mathbb{R}^2 \to \mathbb{R}^2$. Use your answer in (a) to draw a more informative picture of L_A at right. **(1 point)**





4. A matrix $A \in M_{n\times n}(\mathbb{R})$ is called idempotent if $A^2 = A$. Prove that the only possible eigenvalues for an idempotent matrix A are 0 and 1. **(6 points)**

Suppose $v \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue λ, that is, $Av = \lambda v$ with $v \neq 0$.

Then $A^2 v = A(Av) = A(\lambda v) = \lambda(Av)$
$= \lambda(\lambda v) = \lambda^2 v.$

As $A^2 = A$, we also have $A^2 v = Av = \lambda v$.

Combining, get $\lambda^2 v = \lambda v \Rightarrow \lambda(\lambda - 1)v = 0$

As v is an eigenvector, it is not 0, and so we must have $\lambda(\lambda-1) = 0 \Rightarrow \lambda = 0$ or 1.

5. Suppose the only the eigenvalues of $B \in M_{5\times 5}(\mathbb{C})$ are 5 and 7 where $\dim(E_5) = \dim(E_7) = 2$. Determine whether B is diagonalizable and prove your answer. **(6 points)**

: As our field is $\mathbb{C}$, we know that the char poly of B, call it $f(t)$, splits completely.

Hence $f(t) = (5-t)^a(7-t)^b$. As degree $f = 5$, have $a + b = 5$ and so the algebraic multiplicity of one the eigenvalues is at least 3. Since both eigenvalues have geometric multiplicity 2, there is a mismatch between the alg. and geom mult for some eigenvalue and so B is not diagonalizable.

6. On these true/false and short answer questions you **do not need to justify your answers.** **(1 point each)** Ex:29

(a) A square matrix is diagonalizable if and only if its characteristic polynomial splits completely and it has distinct eigenvalues.

True $\boxed{\textbf{False}}$

(b) There exists a transition matrix where 2 is an eigenvalue.

True $\boxed{\textbf{False}}$

(c) Consider $\mathbb{R}^3$ with its usual inner product. If $v \in \mathbb{R}^3$ satisfies $\langle v, e_i \rangle = 0$ for $i = 1, 2, 3$ then $v = 0$.

$\boxed{\textbf{True}}$ **False**

(d) The formula $\langle x, y \rangle = 2x_1y_1 + 3x_2y_2$ defines an inner product on $\mathbb{R}^2$.

$\boxed{\textbf{True}}$ **False**

(e) Give a matrix C in $M_{3\times 3}(\mathbb{R})$ such that $C \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{pmatrix}$.

Take C to be the elementary matrix cor. to $R_2 \leftrightarrow R_1$: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(f) Give a basis for $P_2(\mathbb{C})$ as a vector space over $\mathbb{C}$.

$\{1, x, x^2\}$

(g) Give a basis for $P_2(\mathbb{C})$ as a vector space over $\mathbb{R}$.

$\{1, i, x, ix, x^2, ix^2\}$

(h) The island of Madagascar is home to almost all of the worlds lemurs, a family of primitive non-ape primates. Divide Madagascar into 2 regions, the North and the South, along the Mania river. Suppose that in any given year, 1/3 of the lemurs living in the North move to the South, and 1/6 of those in the South move to the North. Give the transition matrix for the corresponding Markov chain.

$$\begin{array}{cc} N \\ S \end{array} \begin{pmatrix} 2/3 & 1/6 \\ 1/3 & 5/6 \end{pmatrix}$$
$$\quad\; N \quad\;\; S$$

Extra Credit. Suppose T is a linear operator on a finite-dimensional vector space V. If β is a basis for V, define the polynomial $f_\beta(t)$ to be the characteristic polynomial of the matrix $[T]_\beta$. Prove that $f_\beta(t)$ does not depend on the choice of β, that is, $f_\beta(t) = f_\gamma(t)$ for all bases γ of V. **(3 points)** Ex:29

Note that $[T]_\beta$ and $[T]_\gamma$ are similar $n \times n$ matrices as $[T]_\gamma = [I_V]_\beta^\gamma [T]_\beta [I_V]_\gamma^\beta$ and $[I_V]_\beta^\gamma = \left([I_V]_\gamma^\beta\right)^{-1}$. By a homework problem, we know similar matrices have the same characteristic polynomial, and so $f_\beta = f_\gamma$ as needed.