1. Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

(a) Find all the eigenvalues of A. Hint: They are all integers. **(4 points)**

(b) For each eigenvalue find the corresponding eigenspace. **(3 points)**

(c) Diagonalize A, giving both a diagonal matrix D and an invertible matrix Q such that $D = Q^{-1}AQ$. **(3 points)**

2. Suppose $A \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ is a transition matrix associated to a Markov chain.

(a) If $1/2$ and $1/3$ are eigenvalues of A, prove that A is diagonalizable. **(4 points)**

(b) Give a transition matrix B where $1/2$ and $1/3$ are eigenvalues and justify your answer. **(2 points)**

3. The *constant term* of a polynomial $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots a_1 t + a_0$ is the final coefficient a_0.

(a) For the characteristic polynomial of an $n \times n$ matrix A, prove that the constant term is $\det(A)$. Hint: for any polynomial $f(t)$ how does $f(0)$ relate to the constant term of $f(t)$? **(4 points)**

(b) For the characteristic polynomial of an $n \times n$ matrix A, what is the coefficient a_{n-1} of t^{n-1} in terms of the entries of A? **(2 points)**

4. Let V be an inner product space. A linear operator T on V is a *contraction* when $\|T(x)\| < \|x\|$ for all nonzero $x \in V$. Prove that if λ is an eigenvalue for a contraction T then $|\lambda| < 1$. **(5 points)**

5. Suppose $\beta = \{u, v\}$ is an orthonormal basis for an inner product space V over $\mathbb{R}$.

(a) If $x = au + bv$, prove that $\|x\|^2 = a^2 + b^2$. **(4 points)**

(b) For any $x \in V$, prove that $\|x\|^2 = \langle x, u \rangle^2 + \langle x, v \rangle^2$. **(3 points)**

6. Circle the best answer for each part; you do **not** need to provide any justification. **(1 point each)**

(a) If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$ — true / false

(b) The matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is a product of elementary matrices. — true / false

(c) If A and B are similar matrices, then A^3 and B^3 are also similar. — true / false

(d) If all eigenvalues of $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ are 1 then $A = I_n$. — true / false

(e) The vectors $\{(1,0),(i,0),(0,1),(0,i)\}$ are a basis for $\mathbb{C}^2$ regarded as a vector space over $\mathbb{R}$. — true / false

(f) Exactly one of the following limits exist. Circle it.

$$\lim_{n\to\infty}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 0.4 & 0.5 \\ 0.6 & 0.5 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \qquad \lim_{n\to\infty}\begin{pmatrix} 0.2 & 0.3 \\ 0.9 & 0.8 \end{pmatrix}^n$$

Extra Credit. Let $V = P(\mathbb{R})$ be the vector space of polynomials in x of arbitrary degree, that is, the union of all $P_n(\mathbb{R})$. Define an inner product on V by $\langle f(x),\ g(x)\rangle = \int_{-1}^{1} f(x)g(x)dx$. Consider the linear operator T on V given by $T(f(x)) = xf(x)$. Prove or disprove: T is a contraction in the sense of Problem 4. You can continue your answer on the back of this page if needed. **(2 points)**