1. Let \( A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \)

   (a) Find all the eigenvalues of \( A \). Hint: They are all integers.  \( \text{(4 points)} \)

   (b) For each eigenvalue find the corresponding eigenspace. \( \text{(3 points)} \)

   (c) Diagonalize \( A \), giving both a diagonal matrix \( D \) and an invertible matrix \( Q \) such that \( D = Q^{-1}AQ \).  \( \text{(3 points)} \)
2. Suppose $A \in \text{Mat}_{3 \times 3}(\mathbb{R})$ is a transition matrix associated to a Markov chain.

(a) If $1/2$ and $1/3$ are eigenvalues of $A$, prove that $A$ is diagonalizable. \hfill (4 points)

(b) Give a transition matrix $B$ where $1/2$ and $1/3$ are eigenvalues and justify your answer. \hfill (2 points)
3. The constant term of a polynomial \( f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \) is the final coefficient \( a_0 \).

(a) For the characteristic polynomial of an \( n \times n \) matrix \( A \), prove that the constant term is \( \det(A) \). Hint: for any polynomial \( f(t) \) how does \( f(0) \) relate to the constant term of \( f(t) \)? (4 points)

(b) For the characteristic polynomial of an \( n \times n \) matrix \( A \), what is the coefficient \( a_{n-1} \) of \( t^{n-1} \) in terms of the entries of \( A \)? (2 points)
4. Let $V$ be an inner product space. A linear operator $T$ on $V$ is a *contraction* when $\|T(x)\| < \|x\|$ for all nonzero $x \in V$. Prove that if $\lambda$ is an eigenvalue for a contraction $T$ then $|\lambda| < 1$. (5 points)
5. Suppose $\beta = \{u, v\}$ is an orthonormal basis for an inner product space $V$ over $\mathbb{R}$.

(a) If $x = au + bv$, prove that $\|x\|^2 = a^2 + b^2$. (4 points)

(b) For any $x \in V$, prove that $\|x\|^2 = \langle x, u \rangle^2 + \langle x, v \rangle^2$. (3 points)
6. Circle the best answer for each part; you do not need to provide any justification. (1 point each)

(a) If $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = \det(BA)$

(b) The matrix \[
\begin{pmatrix}
1 & 2 \\
2 & 3
\end{pmatrix}
\] is a product of elementary matrices.

(c) If $A$ and $B$ are similar matrices, then $A^3$ and $B^3$ are also similar.

(d) If all eigenvalues of $A \in \text{Mat}_{n \times n}(\mathbb{R})$ are 1 then $A = I_n$.

(e) The vectors \{(1, 0), (i, 0), (0, 1), (0, i)\} are a basis for $\mathbb{C}^2$ regarded as a vector space over $\mathbb{R}$.

(f) Exactly one of the following limits exist. Circle it.
\[
\lim_{n \to \infty} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^n \quad \lim_{n \to \infty} \begin{pmatrix} 0.4 & 0.5 \\ 0.6 & 0.5 \end{pmatrix}^n \quad \lim_{n \to \infty} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \quad \lim_{n \to \infty} \begin{pmatrix} 0.2 & 0.3 \\ 0.9 & 0.8 \end{pmatrix}^n
\]

Extra Credit. Let $V = P(\mathbb{R})$ be the vector space of polynomials in $x$ of arbitrary degree, that is, the union of all $P_n(\mathbb{R})$. Define an inner product on $V$ by $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x) \, dx$. Consider the linear operator $T$ on $V$ given by $T(f(x)) = xf(x)$. Prove or disprove: $T$ is a contraction in the sense of Problem 4. You can continue your answer on the back of this page if needed. (2 points)