1. Consider the linear transformation $T: \mathbb{R}^3 \to P_2(\mathbb{R})$ given by $T(a_1, a_2, a_3) = (a_2 + a_3)x + 3a_1x^2$.

(a) Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and $\gamma = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$. Determine $[T]_\beta^\gamma$. **(2 points)**

$$T(e_1) = T(1,0,0) = 3x^2$$
$$T(e_2) = T(0,1,0) = x$$
$$T(e_3) = T(0,0,1) = x$$

$$\Longrightarrow [T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

(b) Find a basis for the range $\mathcal{R}(T)$. **(4 points)**

As β is a basis for $\mathbb{R}^3$, we know $T(e_1)$, $T(e_2)$ and $T(e_3)$ span $R(T)$. As $T(e_2) = T(e_3) = x$, we see $R(T)$ is spanned by x and $T(e_1) = 3x^2$. As $\beta = \{x, 3x^2\}$ is also lin. indep (since x, $3x^2$ are no colinear), it is a basis for $R(T)$.

(c) Is T an isomorphism? Are $\mathbb{R}^3$ and $P_2(\mathbb{R})$ isomorphic? **(2 points)**

T is not an isomorphism as $R(T) \neq P_2(\mathbb{R})$ as $\dim R(T) = 2$ (by part (b)) whereas $\dim P_2(\mathbb{R}) = 3$. However, $\mathbb{R}^3$ and $P_2(\mathbb{R})$ are isomorphic since they are finite dim'l with the same dimension, namely 3.

9am

1. Consider the linear transformation $T\colon \mathbb{R}^3 \to P_2(\mathbb{R})$ given by $T(a_1, a_2, a_3) = 2a_1x + (a_2 - a_3)x^2$.

(a) Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and $\gamma = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$. Determine $[T]_\beta^\gamma$.

$$\left.\begin{aligned} T(e_1) &= T(1,0,0) = 2x \\ T(e_2) &= T(0,1,0) = x^2 \\ T(e_3) &= T(0,0,1) = -x^2 \end{aligned}\right\} \Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$[T(e_1)]_\gamma$ etc.

(b) Find a basis for the nullspace $\mathcal{N}(T)$.

Have $T(a_1, a_2, a_3) = 0 \Leftrightarrow 2a_1 = 0$ and $a_2 - a_3 = 0$

$\Leftrightarrow a_1 = 0$ and $a_2 = a_3$

So $\mathcal{N}(T) = \{(0,a,a) \mid a \in \mathbb{R}\} = \text{span}(\{(0,1,1)\})$

and so $\beta = \{(0,1,1)\}$ is the desired basis.

(c) Is T an isomorphism?

No since $\mathcal{N}(T) \neq \{0\}$ implies T is not 1-1.

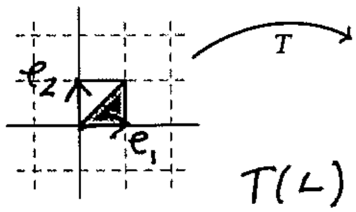
(d) Are $\mathbb{R}^3$ and $P_2(\mathbb{R})$ isomorphic?

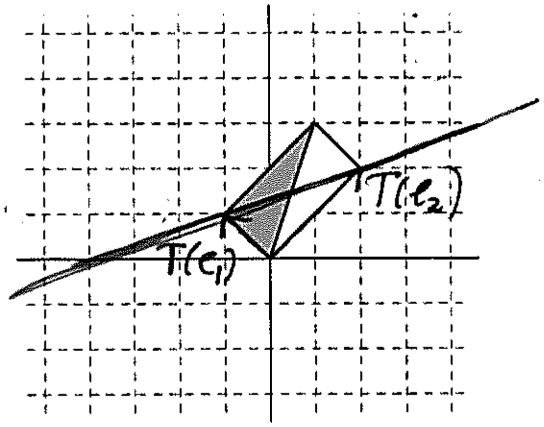
Yes since they are finite dim'l vector spaces of the same dimension, namely 3.

10 am

2. Suppose the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ acts as shown, where the dotted grids consist of unit squares.

(a) For the standard basis $\beta = \{e_1, e_2\}$ of $\mathbb{R}^2$, find $[T]_\beta$. **(3 points)**





From the picture, we see $T(e_1) = (-1, 1)$ and $T(e_2) = (2, 2)$

Thus $[T]_\beta = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$.

(b) Compute $T(3,4)$. **(2 points)**

$$[T(3,4)]_\beta = [T]_\beta [(3,4)]_\beta = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

(c) Find $[T \circ T]_\beta$. **(2 points)**

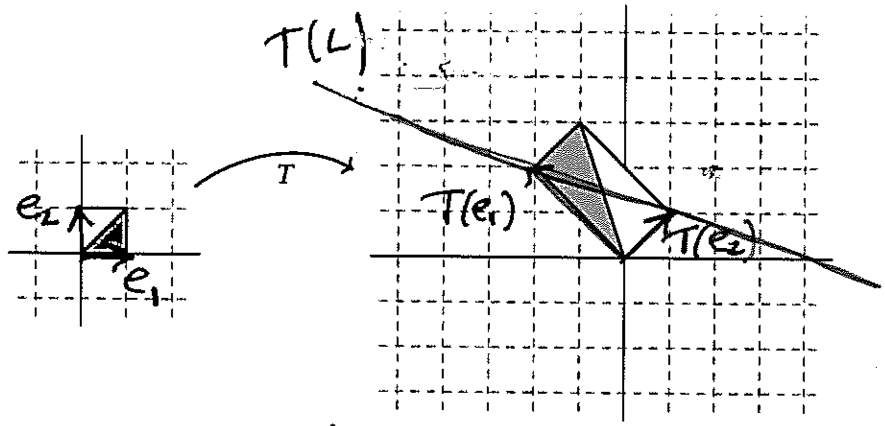
$$[T \circ T]_\beta = [T]_\beta [T_\beta] = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 6 \end{pmatrix}$$

(d) What is the image of the line L given by $x + y = 1$ under T? Explain your reasoning and draw $T(L)$ on the rightmost grid of the picture at the top of this page. **(2 points)**

Since T is linear, it takes lines to lines. The line L contains e_1 and e_2, so the image is the line containing $T(e_1) = (-1,1)$ and $T(e_2) = (2,2)$.

9am

2. Suppose the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ acts as shown, where the dotted grids consist of unit squares.



(a) For the standard basis $\beta = \{e_1, e_2\}$ of $\mathbb{R}^2$, find $[T]_\beta$.

From the picture, we see

$T(e_1) = (-2, 2)$ and

$T(e_2) = (1, 1)$, so $[T]_\beta = \begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix}$

(b) Compute $T(3, 4)$.

We have

$$[T(3,4)]_\beta = [T]_\beta [(3,4)]_\beta = \begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$$

and so $T(3,4) = (-2, 10)$.

(c) Find $[T \circ T]_\beta$.

$$[T \circ T]_\beta = [T]_\beta [T]_\beta = \begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ -2 & 3 \end{pmatrix}$$

(d) What is the image of the line L given by $x + y = 1$ under T? Explain your reasoning and draw your answer on the rightmost grid of the picture at the top of this page.

Since T is linear, it takes lines to lines. The line L contains e_1 and e_2, so the image is the line containing $T(e_1) = (-2, 2)$ and $T(e_2) = (1, 1)$.

10 am

3. Suppose $T\colon \mathbb{R}^3 \to \mathbb{R}$ is linear and there exists $v_0 \in \mathbb{R}^3$ with $T(v_0) = 1$.

(a) Prove that T is onto.

Given $t \in \mathbb{R}$, we have $T(tv_0) = tT(v_0) = t \cdot 1 = t$, and so T is onto. Alternatively, we know $R(T)$ is a subspace of $\mathbb{R}$ and hence either $\{0\}$ or $\mathbb{R}$; as $T(v_0) \neq 0$, must have $R(T) = \mathbb{R}$, i.e. T is onto.

(b) Find $\dim \mathcal{N}(T)$.

By the Dimension Theorem,

$$\dim \mathcal{N}(T) = \dim \mathbb{R}^3 - \dim \underbrace{R(T)}_{= \mathbb{R} \text{ by (a)}} = 3 - 1 = 2.$$

(c) Let $V = \{v \in \mathbb{R}^3 \mid T(v) = 1\}$ and consider $W = \{v_0 + a \mid a \in \mathcal{N}(T)\}$. Prove that $V = W$.

First, we show $V \supseteq W$. Given $w = v_0 + a$ in W, we have $T(w) = T(v_0) + T(a) = 1 + 0 = 1$ since $a \in \mathcal{N}(T)$. Thus $w \in V$ as desired. Conversely, suppose $v \in V$ and set $a = v - v_0$. Then $T(a) = T(v) - T(v_0)$ $= 1 - 1 = 0$ since $v \in V$. Therefore $a \in \mathcal{N}(T)$ and $v = v_0 + a$, proving $v \in W$. Thus $V \subseteq W$ as well, and so $V = W$ as desired.

(d) What do $\mathcal{N}(T)$ and V represent geometrically in $\mathbb{R}^3$ and how do they relate to each other?

They are a pair of parallel planes.

4. Define an $n \times n$ matrix A to be *annoying* when $A^2 = 0$. (This terminology is nonstandard.)

(a) Circle the unique matrix below that is annoying. **(1 point)**

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

Reason: $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(b) Prove or give a counterexample: Annoying matrices are not invertible. **(2 points)**

Proof: Assume A is invertible. As $A^2 = AA = 0$,

we have $0 = (A^{-1}A^{-1})0 = (A^{-1}A^{-1})(AA)$

$= A^{-1}(A^{-1}A)A = A^{-1}I_n A = A^{-1}A = I_n$

which is a contradiction. So A must not be invertible.

(c) If A is annoying, prove that $B = I_n + A$ is invertible with inverse $C = I_n - A$. **(3 points)**

Since $BC = (I+A)(I-A) = (I+A)I + (I+A)(-A)$

$= I + A - A + \underbrace{A^2}_{=0} = I$

and $CB = (I-A)(I+A) = I - A + A - A^2 = I$

we have that B is invertible with inverse C.

5. (a) Let $A = \begin{pmatrix} 3 & 3 & 6 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}$. For the linear transformation $L_A\colon \mathbb{R}^3 \to \mathbb{R}^3$, compute $L_A(2,0,1)$. **(1 point)**

$$L_A(2,0,1) = \begin{pmatrix} 3 & 3 & 6 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -3 \end{pmatrix}$$

(b) Compute $\det(A)$ by cofactor expansion. **(2 points)**

Expanding on the 2^{nd} row gives

$$\det(A) = (-1)^{2+2} \cdot 1 \cdot \det\begin{pmatrix} 3 & 6 \\ -2 & 1 \end{pmatrix} = 3 + 12 = 15$$

(c) Compute $\det(A)$ by a different method that involves row operations. **(2 points)**

$$A \xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix} \xrightarrow{2R_1 + R_3} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} = B$$

As B is upper triangular, have $\det(B) = 1 \cdot 1 \cdot 5 = 5$.
Only the 1^{st} row op changes the det, specifically $\det(B) = \frac{1}{3}\det(A)$. Solving for $\det(A)$, we get

$$\det(A) = 3\det(B) = 3 \cdot 5 = 15.$$

9 am

5. (a) Let $A = \begin{pmatrix} 3 & 3 & 6 \\ 0 & 1 & 0 \\ -2 & -2 & 2 \end{pmatrix}$. For the linear transformation $L_A \colon \mathbb{R}^3 \to \mathbb{R}^3$, compute $L_A(1,0,2)$. **(1 point)**

$$L_A(1,0,2) = \begin{pmatrix} 3 & 3 & 6 \\ 0 & 1 & 0 \\ -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \\ 2 \end{pmatrix}$$

(b) Compute $\det(A)$ by cofactor expansion. **(2 points)**

Expanding on the 2nd row gives

$$\det(A) = (-1)^{2+2} \cdot 1 \cdot \det\begin{pmatrix} 3 & 6 \\ -2 & 2 \end{pmatrix} = 6 + 12 = 18$$

(c) Compute $\det(A)$ by a different method that involves row operations. **(2 points)**

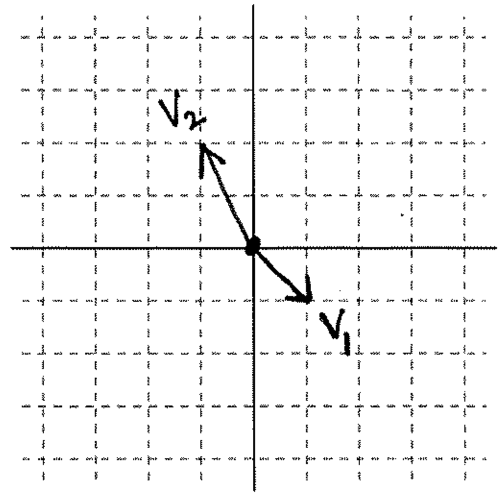
$$A \xrightarrow{1/3 R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ -2 & -2 & 2 \end{pmatrix} \xrightarrow{2R_1 + R_3} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} = B$$

As B is upper triangular, have $\det B = 1 \cdot 1 \cdot 6 = 6$. Only the first row op changes the det, so $\det(B) = \frac{1}{3}\det(A)$. Solving for $\det(A)$ we get

$$\det(A) = 3\det(B) = 3 \cdot 6 = 18.$$

10 am

6. Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. Suppose $\gamma = \{v_1, v_2\}$ is the basis where $[I_{\mathbb{R}^2}]_\beta^\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Draw and label the vectors v_1 and v_2 on the grid made of unit squares at right. **(3 points)**



$$[I_{\mathbb{R}^2}]_\gamma^\beta = \left([I_{\mathbb{R}}]_\beta^\gamma\right)^{-1} = \frac{1}{2\cdot 1 - 1\cdot 1}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

So $[v_1]_\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $[v_2]_\beta = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

7. Circle true or false as appropriate; you do **not** need to provide any justification. **(1 point each)**

(a) Suppose S and T are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^3$. If $S(1,1) = T(1,1)$ and $S(0,2) = T(0,2)$ then $S = T$.

true false

(b) There is a 2×2 matrix A such that A is not invertible but the linear transformation $L_A \colon \mathbb{R}^2 \to \mathbb{R}^2$ is invertible.

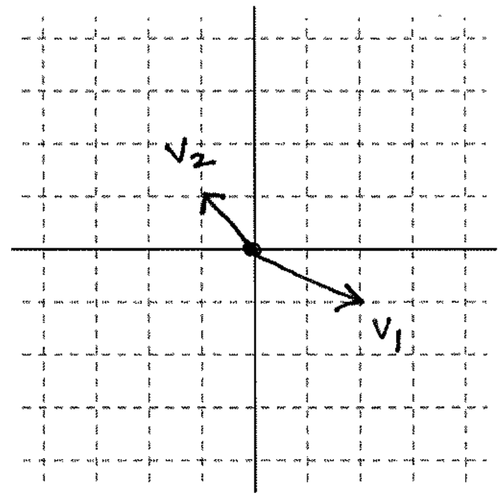
true false

(c) Suppose $T \colon V \to W$ is a linear transformation. If vectors $\{v_1, v_2, \ldots, v_n\}$ in V have the property that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ span W, then $\{v_1, v_2, \ldots, v_n\}$ span V.

true false

9am

6. Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. Suppose $\gamma = \{v_1, v_2\}$ is the basis where $[I_{\mathbb{R}^2}]_\beta^\gamma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Draw and label the vectors v_1 and v_2 on the grid made of unit squares at right. **(3 points)**



$$[I_{\mathbb{R}^2}]_\gamma^\beta = \left([I_{\mathbb{R}^2}]_\beta^\gamma\right)^{-1} = \frac{1}{2-1}\cdot\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Hence $[v_1]_\beta = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and

$[v_2]_\beta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

7. Circle true or false as appropriate; you do **not** need to provide any justification. **(1 point each)**

(a) Suppose $T\colon V \to W$ is a linear transformation. If vectors $\{v_1, v_2, \ldots, v_n\}$ in V have the property that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ span W, then $\{v_1, v_2, \ldots, v_n\}$ span V. | true | false |

(b) There is a 2×2 matrix A such that A is invertible but the linear transformation $L_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is not invertible. | true | false |

(c) Suppose S and T are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^3$. If $S(1,1) = T(1,1)$ and $S(0,2) = T(0,2)$ then $S = T$. | true | false |

10 am