

There are 45 points possible on this exam. Take care to note that the problems are not weighted equally. Calculators, phones, books, notes (beyond the allowed one page) and suchlike aids to gracious living are not permitted. Show all your work as credit will not be given for correct answers without proper justification, except where indicated in Problem 6.

Do not start until instructed.

Good luck!



1. Consider the left-multiplication linear transformation $L_A: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ where $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$.

(a) Compute $L_A(-1, 1)$. **(1 point)**

(b) Find a basis for $\mathcal{N}(L_A)$. **(3 points)**

(c) Find a basis for $\mathcal{R}(L_A)$. **(3 points)**

(d) Verify directly that the Dimension Theorem holds for L_A . **(2 points)**

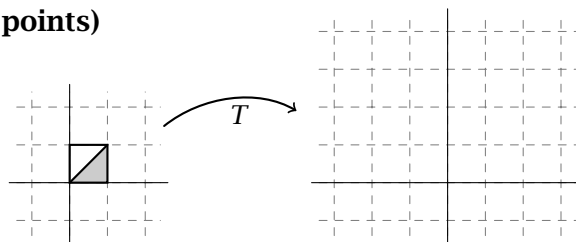
(e) Let $\beta = \{e_1, e_2\}$ and $\gamma = \{e_1, e_2, e_3\}$ be the standard bases for \mathbb{R}^2 and \mathbb{R}^3 respectively. What is $[L_A]_{\beta}^{\gamma}$? **(1 point)**



2. Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $T(x, y) = (-x + 2y, 2x + y)$.

Ex:0

(a) Complete the picture at right to visually describe T . **(2 points)**



(b) Let $\beta = \{e_1, e_2\}$ be the standard basis for \mathbb{R}^2 . Compute $[T]_\beta$. **(1 point)**

(c) Consider the basis $\gamma = \{v_1 = (1, 1), v_2 = (1, 3)\}$ for \mathbb{R}^2 . Find the change of basis matrices $[I_{\mathbb{R}^2}]_\beta^\gamma$ and $[I_{\mathbb{R}^2}]_\gamma^\beta$. **(3 points)**

(d) Use your answers in (b) and (c) to compute $[T]_\gamma$. **(3 points)**

(e) Let $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ be the closed disc bounded by the unit circle about the origin. Calculate $\text{Area}(T(D))$. **(1 point)**



3. Suppose $T: V \rightarrow W$ is a linear transformation between finite-dimensional vector spaces over \mathbb{R} . Ex:0

(a) Suppose V_0 is a subspace of V . Prove that $W_0 = T(V_0)$ is a subspace of W . **(5 points)**

(b) If T is an isomorphism, prove that $\dim(V_0) = \dim(W_0)$. **(3 points)**



4. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, compute $\det(A)$ by expanding along the *second row*. **(4 points)** .

5. Suppose $A \in M_{n \times n}(\mathbb{R})$.

(a) Prove that $\text{rank}(A) = \text{rank}(A^t)$. **(4 points)**

(b) If A is invertible, prove that A^t is invertible. Additionally, find and prove a relationship between $(A^t)^{-1}$ and A^{-1} . **(3 points)**



6. For this problem, you do *not* need to justify your answers. (1 point each)

Ex:0

(a) The vector spaces $M_{2 \times 2}(\mathbb{R})$ and $P_4(\mathbb{R})$ are isomorphic.

True **False**

(b) Suppose $T: V \rightarrow W$ is a function between two vector spaces where $T(v_1 + cv_2) = T(v_1) + cT(v_2)$ for all $v_1, v_2 \in V$ and $c \in \mathbb{R}$. Then T is linear.

True **False**

(c) For all square matrices A , one has $\det(-A) = -\det(A)$.

True **False**

(d) Suppose $T: V \rightarrow W$ is linear and onto. If $v_1, v_2 \in V$ with $T(v_2) = 3T(v_1)$ then $v_2 = 3v_1$.

True **False**

(e) For all $A \in M_{n \times n}(\mathbb{R})$, one has $\mathcal{R}(L_A) = \text{ColumnSpace}(A)$.

True **False**

(f) If vector spaces V and W are isomorphic, then there is exactly one isomorphism $T: V \rightarrow W$ between them.

True **False**

Extra Credit. Consider the line $L = \text{span}\{(1, 1, 1)\}$ in \mathbb{R}^3 . Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rigid rotation about the line L through angle π . Find the matrix of the linear transformation T with respect to the standard basis of \mathbb{R}^3 . (If you need more space, use the back of this sheet.) (3 points) .

