1. Consider the left-multiplication linear transformation $L_A\colon \mathbb{R}^2 \to \mathbb{R}^3$ where $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$.

(a) Compute $L_A(-1,1)$. **(1 point)**

$$L_A(-1,1) = A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(b) Find a basis for $\mathcal{N}(L_A)$. **(3 points)** First, $\mathcal{N}(L_A) = \{x \in \mathbb{R}^2 \mid L_A(x) = 0\}$ $= \{x \in \mathbb{R}^2 \mid Ax = 0\} = \mathcal{N}(A)$. As $\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$ row reduces to $\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ via $-2R_1 + R_2$ and $-3R_1 + R_2$, we see

$\mathcal{N}(A) = \{(-2t, t) \mid t \in \mathbb{R}\} = \operatorname{span}\{(-2,1)\}$

and so the basis we seek is $\{(-2,1)\}$.

(c) Find a basis for $\mathcal{R}(L_A)$. **(3 points)**

As $\mathcal{R}(L_A) = \operatorname{ColSp}(A)$ and the second column is twice the first, we have

$$\operatorname{ColSp}(A) = \operatorname{span}\{(1,2,3)\}$$

and so the requested basis is $\{(1,2,3)\}$

(d) Verify directly that the Dimension Theorem holds for L_A. **(2 points)**

We have $\dim(\mathcal{N}(L_A)) + \dim(\mathcal{R}(L_A)) = 1 + 1 = 2$

which does match $\dim\begin{pmatrix} \text{domain} \\ \text{of } L_A \end{pmatrix} = 2$.

(e) Let $\beta = \{e_1, e_2\}$ and $\gamma = \{e_1, e_2, e_3\}$ be the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. What is $[L_A]_\beta^\gamma$? **(1 point)**

As β and γ are standard, we have

$$[L_A]_\beta^\gamma = A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}.$$

1. Consider the left-multiplication linear transformation $L_A\colon \mathbb{R}^2 \to \mathbb{R}^3$ where $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{pmatrix}$.

(a) Compute $L_A(-1,1)$. **(1 point)**

$$L_A(-1,1) = A\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2&1\\4&2\\6&3\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-3\end{pmatrix} = (-1,-2,-3).$$

(b) Find a basis for $\mathcal{N}(L_A)$. **(3 points)** As we have

$\mathcal{N}(L_A) = \mathcal{N}(A)$, it suffices to row reduce A

$$\xrightarrow{-2R_1+R_2} \begin{pmatrix}2&1\\0&0\\6&3\end{pmatrix} \xrightarrow{-3R_1+R_3} \begin{pmatrix}2&1\\0&0\\0&0\end{pmatrix}$$ to see that

$\mathcal{N}(L_A) = \{(t,-2t) \mid t \in \mathbb{R}\}$ which has basis $\beta = \{(-1,2)\}$.

(c) Find a basis for $\mathcal{R}(L_A)$. **(3 points)**

Have $\mathcal{R}(L_A) = \mathrm{ColSp}(A)$ and as the 1^{st} column is twice the second one, we see that

$$\mathcal{R}(L_A) = \mathrm{span}\{(1,2,3)\}$$

and so $\{(1,2,3)\}$ is the basis we seek.

(d) Verify directly that the Dimension Theorem holds for L_A. **(2 points)** We have

$$\dim(\mathcal{N}(L_A)) + \dim(\mathcal{R}(L_A)) = 1 + 1 = 2$$

which matches $\dim\left(\text{domain of } L_A = \mathbb{R}^2\right) = 2$.

(e) Let $\beta = \{e_1, e_2\}$ and $\gamma = \{e_1, e_2, e_3\}$ be the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. What is $[L_A]_\beta^\gamma$? **(1 point)**

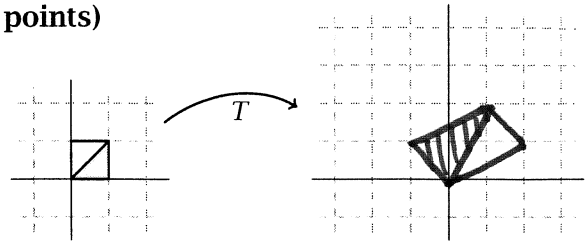
It's just A itself, i.e. $[L_A]_\beta^\gamma = \begin{pmatrix}2&1\\4&2\\6&3\end{pmatrix}$

2. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (-x+2y, x+y)$. Ex:68

(a) Complete the picture at right to visually describe T. **(2 points)**

$T(e_1) = T(1,0) = (-1, 1)$

$T(e_2) = T(0,1) = (2, 1)$



(b) Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. Compute $[T]_\beta$. **(1 point)**

$$[T]_\beta = \begin{bmatrix} [T(e_1)]_\beta & [T(e_2)]_\beta \end{bmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

(c) Consider the basis $\gamma = \{v_1 = (1,-1), v_2 = (1,2)\}$ for $\mathbb{R}^2$. Find the change of basis matrices $[I_{\mathbb{R}^2}]_\beta^\gamma$ and $[I_{\mathbb{R}^2}]_\gamma^\beta$. **(3 points)**

Have $[I]_\gamma^\beta = \begin{pmatrix} [v_1]_\beta & [v_2]_\beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$. Then

$$[I]_\beta^\gamma = \left([I]_\gamma^\beta\right)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

(d) Use your answers in (b) and (c) to compute $[T]_\gamma$. **(3 points)**

$$[T]_\gamma = [I]_\beta^\gamma [T]_\beta [I]_\gamma^\beta = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -3 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$$

(e) Let $D = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; x_1^2 + x_2^2 \le 1 \right\}$ be the closed disc bounded by the unit circle about the origin. Calculate $\text{Area}(T(D))$. **(1 point)**

Have $T = L_A$ for $A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$ and know from class that L_A changes areas by $|\det(A)| = |-3| = 3$

So as $\text{Area}(D) = \pi 1^2 = \pi$, have $\text{Area}(T(D)) = 3\pi$.

3. Suppose $T\colon V \to W$ is a linear transformation between finite-dimensional vector spaces over $\mathbb{R}$. Ex:68

(a) Suppose V_0 is a subspace of V. Prove that $W_0 = T(V_0)$ is a subspace of W. **(5 points)**

1) As V_0 is a subspace, $0_V \in V_0$. As T is linear, $T(0_V) = 0_W$. Thus $0_W \in W_0$ as needed.

2) Suppose $w_1, w_2 \in W_0$. Then, there are $v_1, v_2 \in V_0$ with $T(v_i) = w_i$. As T is linear

$$T(v_1+v_2) = T(v_1)+T(v_2) = w_1+w_2$$

and as V_0 is a subspace we have $v_1+v_2 \in V_0$. Thus w_1+w_2 is in W_0, so W_0 is closed under addition.

3) Suppose $w_1 \in W_0$, $c \in \mathbb{R}$. Choose $v_1 \in V_0$ with $T(v_1) = w_1$. As V_0 is a subspace + T is linear, we have $cv_1 \in V_0$ and $T(cv_1) = cT(v_1) = cw_1$. Hence $cw_1 \in W_0$ and W_0 is closed under scalar mult.

(b) If T is an isomorphism, prove that $\dim(V_0) = \dim(W_0)$. **(3 points)**

Let $T' : V_0 \to W_0$ be the restriction of T to V_0, i.e. $T'(v) = T(v)$ for all $v \in V_0$ but T' only accepts inputs from V_0.

As $T: V \to W$ is 1-1 and onto, we have that $T': V_0 \to W_0$ is also 1-1 and onto since $W_0 = T(V_0)$. As T' is also linear, we've learned that T' is an isomorphism of vector spaces. Hence $\dim V_0 = \dim W_0$

4. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, compute $\det(A)$ by expanding along the *second row*. **(4 points)** .

$$\det A = \sum_{j=1}^{3} (-1)^{j+2} A_{2j} \det(\tilde{A}_{2j})$$

$$= -2 \det\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} + (-1)\cdot \det\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} + 0 \cdot \det(?)$$

$$= -2 \cdot (-4) - 1 \cdot (-8) = 16$$

5. Suppose $A \in M_{n\times n}(\mathbb{R})$.

(a) Prove that $\operatorname{rank}(A) = \operatorname{rank}(A^t)$. **(4 points)** For any matrix B, we have $\operatorname{rank}(B) \underset{\text{def}}{=} \dim(\operatorname{ColSp}(B)) \underset{\text{Thm}}{=} \dim(\operatorname{RowSp}(B))$

As the rows of A^t are the columns of A, we have $\operatorname{RowSp}(A^t) = \operatorname{ColSp}(A)$. ~~Taking the common~~ Consequently, $\operatorname{rank}(A^t) = \operatorname{rank}(A)$.

(b) If A is invertible, prove that A^t is invertible. Additionally, find and prove a relationship between $(A^t)^{-1}$ and A^{-1}. **(3 points)**

If A is invertible, then $\operatorname{rank}(A) = n \Rightarrow \operatorname{rank}(A^t) = n \Rightarrow A^t$ is invertible. In addition, I claim $B = (A^{-1})^t$ is $(A^t)^{-1}$. This follows as

$$A^t \cdot B = A^t (A^{-1})^t = (A^{-1} \cdot A)^t = I_n^t = I_n$$

and $B \cdot A^t = (A^{-1})^t A^t = (A A^{-1})^t = I_n^t = I_n$.

4. For $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, compute $\det(A)$ by expanding along the *second row.* **(4 points)** .

$$\det A = \sum_{j=1}^{3} (-1)^{j+2} A_{2j} \det(\tilde{A}_{2j})$$

$$= -2 \cdot \det\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} + (-1) \det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + 0 \cdot \det(?)$$

$$= -2(-2) + (-1)(-5) = 9.$$

5. Suppose $A \in M_{n \times n}(\mathbb{R})$.

(a) Prove that $\operatorname{rank}(A) = \operatorname{rank}(A^t)$. **(4 points)**

See previous page.

(b) If A is invertible, prove that A^t is invertible. Additionally, find and prove a relationship between $(A^t)^{-1}$ and A^{-1}. **(3 points)**

6. For this problem, you do *not* need to justify your answers. **(1 point each)**

(a) The vector spaces $M_{2\times 2}(\mathbb{R})$ and $P_4(\mathbb{R})$ are isomorphic.

True **False**

(b) Suppose $T\colon V \to W$ is a function between two vector spaces where $T(v_1 + cv_2) = T(v_1) + cT(v_2)$ for all $v_1, v_2 \in V$ and $c \in \mathbb{R}$. Then T is linear.

True **False**

(c) For all square matrices A, one has $\det(-A) = -\det(A)$.

True **False**

(d) Suppose $T\colon V \to W$ is linear and onto. If $v_1, v_2 \in V$ with $T(v_2) = 3T(v_1)$ then $v_2 = 3v_1$.

True **False**

(e) For all $A \in M_{n\times n}(\mathbb{R})$, one has $\mathcal{R}(L_A) = \text{ColumnSpace(A)}$.

True **False**

(f) If vector spaces V and W are isomorphic, then there is exactly one isomorphism $T\colon V \to W$ between them.

True **False**

Extra Credit. Consider the line $L = \text{span}\{(1,1,1)\}$ in $\mathbb{R}^3$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rigid rotation about the line L through angle π. Find the matrix of the linear transformation T with respect to the standard basis of $\mathbb{R}^3$. (If you need more space, use the back of this sheet.) **(3 points)**.

See next page.

6. For this problem, you do *not* need to justify your answers. **(1 point each)** Ex:67

(a) The vector spaces $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$ are isomorphic.

True **False**

(b) For all square matrices A, one has $\det(-A) = -\det(A)$.

True **False**

(c) Suppose $T\colon V \to W$ is a function between two vector spaces where $T(v_1 + cv_2) = T(v_1) + cT(v_2)$ for all $v_1, v_2 \in V$ and $c \in \mathbb{R}$. Then T is linear.

True **False**

(d) Suppose $T\colon V \to W$ is linear and 1-1. If $v_1, v_2 \in V$ with $T(v_2) = 3T(v_1)$ then $v_2 = 3v_1$.

True **False**

(e) For all $A \in M_{n\times n}(\mathbb{R})$, one has $\mathcal{R}(L_A) = \text{RowSpace(A)}$.

True **False**

(f) If vector spaces V and W are isomorphic, then there is exactly one isomorphism $T\colon V \to W$ between them.

True **False**

Extra Credit. Consider the line $L = \text{span}\{(1,1,1)\}$ in $\mathbb{R}^3$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rigid rotation about the line L through angle π. Find the matrix of the linear transformation T with respect to the standard basis of $\mathbb{R}^3$. (If you need more space, use the back of this sheet.) **(3 points)**.

Let $v_1 = (1,1,1)$, $v_2 = (1,-1,0)$, $v_3 = (0,1,-1)$ so that $\gamma = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$. Since $v_1 \in L$ and v_2 and v_3 are perpendicular to L (as $v_1 \cdot v_2 = v_1 \cdot v_3 = 0$) we have $T(v_1) = v_1$ and $T(v_2) = -v_2$ and $T(v_3) = -v_3$ so that $[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. If β is the standard basis, we have $Q = [I_{\mathbb{R}^3}]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$, and $Q^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$.

Thus
$$[T]_\beta = [I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma = Q\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}Q^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$