1. Consider the linear transformation $T\colon \mathbb{R}^3 \to P_2(\mathbb{R})$ given by $T(a_1, a_2, a_3) = (a_2 + a_3)x + 3a_1x^2$.

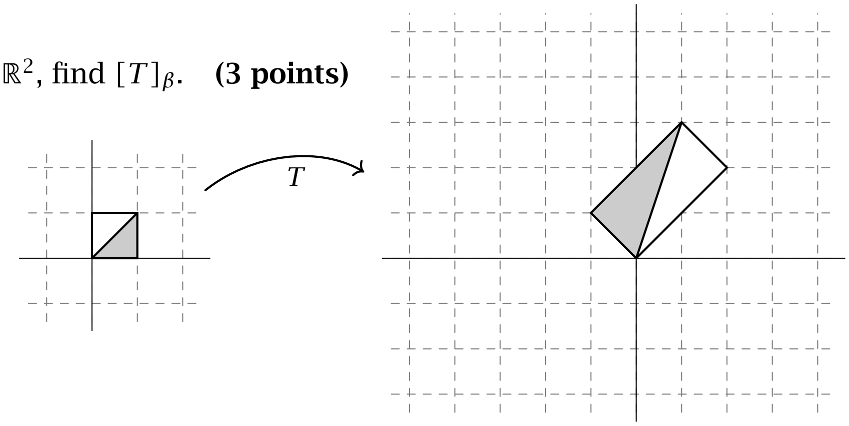
(a) Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and $\gamma = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$. Determine $[T]_\beta^\gamma$. **(2 points)**

(b) Find a basis for the range $\mathcal{R}(T)$. **(4 points)**

(c) Is T an isomorphism? Are $\mathbb{R}^3$ and $P_2(\mathbb{R})$ isomorphic? **(2 points)**

2. Suppose the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ acts as shown, where the dotted grids consist of unit squares.

(a) For the standard basis $\beta = \{e_1, e_2\}$ of $\mathbb{R}^2$, find $[T]_\beta$. **(3 points)**



(b) Compute $T(3,4)$. **(2 points)**

(c) Find $[T \circ T]_\beta$. **(2 points)**

(d) What is the image of the line L given by $x + y = 1$ under T? Explain your reasoning and draw $T(L)$ on the rightmost grid of the picture at the top of this page. **(2 points)**

3. Suppose $T\colon \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation and there exists $v_0 \in \mathbb{R}^3$ with $T(v_0) = 1$.

(a) Prove that T is onto. **(2 points)**

(b) Find $\dim \mathcal{N}(T)$. **(2 points)**

(c) Let $V = \{v \in \mathbb{R}^3 \mid T(v) = 1\}$ and consider $W = \{v_0 + a \mid a \in \mathcal{N}(T)\}$. Prove that $V = W$. **(4 points)**

(d) What do $\mathcal{N}(T)$ and V represent geometrically in $\mathbb{R}^3$ and how do they relate to each other? **(2 points)**

4. Define an $n \times n$ matrix A to be *annoying* when $A^2 = 0$. (This terminology is nonstandard.)

(a) Circle the unique matrix below that is annoying. **(1 point)**

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

(b) Prove or give a counterexample: Annoying matrices are not invertible. **(2 points)**

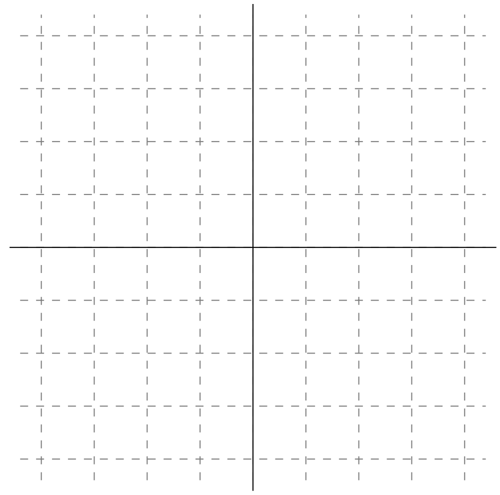
(c) If A is annoying, prove that $B = I_n + A$ is invertible with inverse $C = I_n - A$. **(3 points)**

5. (a) Let $A = \begin{pmatrix} 3 & 3 & 6 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}$. For the linear transformation $L_A\colon \mathbb{R}^3 \to \mathbb{R}^3$, compute $L_A(2,0,1)$. **(1 point)**

(b) Compute $\det(A)$ by cofactor expansion. **(2 points)**

(c) Compute $\det(A)$ by a different method that involves row operations. **(2 points)**

6. Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. Suppose $\gamma = \{v_1, v_2\}$ is the basis where $[I_{\mathbb{R}^2}]_\beta^\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Draw and label the vectors v_1 and v_2 on the grid made of unit squares at right. **(3 points)**



7. Circle true or false as appropriate; you do **not** need to provide any justification. **(1 point each)**

(a) Suppose S and T are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^3$. If $S(1,1) = T(1,1)$ and $S(0,2) = T(0,2)$ then $S = T$.

true false

(b) There is a 2×2 matrix A such that A is not invertible but the linear transformation $L_A \colon \mathbb{R}^2 \to \mathbb{R}^2$ is invertible.

true false

(c) Suppose $T \colon V \to W$ is a linear transformation. If vectors $\{v_1, v_2, \ldots, v_n\}$ in V have the property that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ span W, then $\{v_1, v_2, \ldots, v_n\}$ span V.

true false