1. Consider the linear transformation \( T : \mathbb{R}^3 \rightarrow P_2(\mathbb{R}) \) given by \( T(a_1, a_2, a_3) = (a_2 + a_3)x + 3a_1x^2 \).

(a) Let \( \beta = \{e_1, e_2, e_3\} \) be the standard basis for \( \mathbb{R}^3 \) and \( \gamma = \{1, x, x^2\} \) be the standard basis for \( P_2(\mathbb{R}) \). Determine \([T]_\beta^\gamma\).  \( \textbf{(2 points)} \)

(b) Find a basis for the range \( \mathcal{R}(T) \).  \( \textbf{(4 points)} \)

(c) Is \( T \) an isomorphism? Are \( \mathbb{R}^3 \) and \( P_2(\mathbb{R}) \) isomorphic?  \( \textbf{(2 points)} \)
2. Suppose the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ acts as shown, where the dotted grids consist of unit squares.

(a) For the standard basis $\beta = \{e_1, e_2\}$ of $\mathbb{R}^2$, find $[T]_\beta$. (3 points)

(b) Compute $T(3, 4)$. (2 points)

(c) Find $[T \circ T]_\beta$. (2 points)

(d) What is the image of the line $L$ given by $x + y = 1$ under $T$? Explain your reasoning and draw $T(L)$ on the rightmost grid of the picture at the top of this page. (2 points)
3. Suppose $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a linear transformation and there exists $v_0 \in \mathbb{R}^3$ with $T(v_0) = 1$.

(a) Prove that $T$ is onto. (2 points)

(b) Find $\text{dim} \mathcal{N}(T)$. (2 points)

(c) Let $V = \{ v \in \mathbb{R}^3 \mid T(v) = 1 \}$ and consider $W = \{ v_0 + a \mid a \in \mathcal{N}(T) \}$. Prove that $V = W$. (4 points)

(d) What do $\mathcal{N}(T)$ and $V$ represent geometrically in $\mathbb{R}^3$ and how do they relate to each other? (2 points)
4. Define an $n \times n$ matrix $A$ to be annoying when $A^2 = 0$. (This terminology is nonstandard.)

(a) Circle the unique matrix below that is annoying. (1 point)

\[
\begin{bmatrix}
  1 & 2 \\
  0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
  2 & -1 \\
  4 & -2
\end{bmatrix}, \quad
\begin{bmatrix}
  1 & 2 \\
  -3 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
  3 & 0 \\
  0 & 0
\end{bmatrix}
\]

(b) Prove or give a counterexample: Annoying matrices are not invertible. (2 points)

(c) If $A$ is annoying, prove that $B = I_n + A$ is invertible with inverse $C = I_n - A$. (3 points)
5. (a) Let \( A = \begin{bmatrix} 3 & 3 & 6 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} \). For the linear transformation \( L_A : \mathbb{R}^3 \to \mathbb{R}^3 \), compute \( L_A(2, 0, 1) \). \( \text{(1 point)} \)

(b) Compute \( \det(A) \) by cofactor expansion. \( \text{(2 points)} \)

(c) Compute \( \det(A) \) by a different method that involves row operations. \( \text{(2 points)} \)
6. Let \( \beta = \{e_1, e_2\} \) be the standard basis for \( \mathbb{R}^2 \). Suppose \( \gamma = \{v_1, v_2\} \) is the basis where \( [I_{\mathbb{R}^2}]_{\beta}^{\gamma} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \). Draw and label the vectors \( v_1 \) and \( v_2 \) on the grid made of unit squares at right. (3 points)

7. Circle true or false as appropriate; you do not need to provide any justification. (1 point each)

(a) Suppose \( S \) and \( T \) are linear transformations from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \). If \( S(1, 1) = T(1, 1) \) and \( S(0, 2) = T(0, 2) \) then \( S = T \).

true false

(b) There is a \( 2 \times 2 \) matrix \( A \) such that \( A \) is not invertible but the linear transformation \( L_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is invertible.

true false

(c) Suppose \( T: V \rightarrow W \) is a linear transformation. If vectors \( \{v_1, v_2, \ldots, v_n\} \) in \( V \) have the property that \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \) span \( W \), then \( \{v_1, v_2, \ldots, v_n\} \) span \( V \).

true false