**1.** Consider the matrix 
$$A = \begin{pmatrix} 3 & 1 & -5 & 0 & 5 \\ 2 & 1 & -3 & 1 & 7 \\ 1 & 1 & -1 & 1 & 6 \end{pmatrix}$$
 which is row equivalent to  $B = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$ .

(a) What is the relationship between the solution sets to  $\mathcal{LS}(A)$  and  $\mathcal{LS}(B)$ ? (1 point)

As A and B are row equivalent, the two solution sets are the same

(b) Find a matrix *C* that is row equivalent to *A* and is in reduced row echelon form. Label any row operations you preform. **(2 points)** 

A my B my 
$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} = C$$

(c) Parameterize all solutions to the linear system LS(A). (3 points) This is the same as solving ZS(C) which has one free variable corr. to the  $3^{rd}$  column. Our lyns are

 $\chi_1 - 2\chi_3 = 1$  $\chi_2 + \chi_3 = 2$  and hence our param

$$\chi_4 = 3$$

Solutions are {(1+2t, 2-t, t, 3) | tell.

(d) Find the dimension of the rowspace  $\mathcal{R}(A)$  of A. (2 points)

Know R(A) = R(C). As C is in RREF, a basis for C is the set of all nonzero rows. So dim R(A) = 3.

9 am version

**1.** Consider the matrix 
$$A = \begin{pmatrix} 3 & 1 & -5 & 0 & 5 \\ 2 & 1 & -3 & 1 & 7 \\ 1 & 1 & -1 & 1 & 6 \end{pmatrix}$$
 which is row equivalent to  $B = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$ .

(a) What is the relationship between the solution sets to  $\mathcal{LS}(A)$  and  $\mathcal{LS}(B)$ ? (1 point)

As A and B are row equivalent, the two solution sets are the same

(b) Find a matrix *C* that is row equivalent to *A* and is in reduced row echelon form. Label any row operations you preform. **(2 points)** 

$$A \xrightarrow{\text{row}} B \xrightarrow{-R_3 + R_2} \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} = C$$

(c) Parameterize all solutions to the linear system LS(A). (3 points) This is the Same as solving ZS(C) which has one free variable corr. to the  $3^{rd}$  column. Our egns are  $\chi_1 -2\chi_3 = 1$   $\chi_2 + \chi_3 = 2$  and hence our param.  $\chi_4 = 3$ 

solutions are {(1+2t, 2-t, t, 3) | t∈R}

(d) Find the dimension of the nullspace  $\mathcal{N}(A)$  of A. (2 points)

Know N(A) = N(C) = solins to ZS(C::).

As Cisin RREF, dim N(c) = # of non pivot = 2

So  $\dim \mathcal{N}(A) = 2$  as well.

10 am version

- **2.** Let *W* be a subset of a vector space *V*. We say that *W* is *awesome* when both:
  - (i) The subset W is nonempty.
  - (ii) For all  $w_1$  and  $w_2$  in W and  $a \in \mathbb{R}$ , the linear combination  $aw_1 + w_2$  is also in W.
  - (a) Prove that if W is a subspace then it is awesome.

As Wis a subspace, it contains the O vector; in particular, it is nonempty and so satisfies (i).

As a subspace, Wis closed under addition and scalar mult. So if w, we wand a & R we have aw, & W and hence aw, + we is in W.

So W has (ii) as well and hence is a we some.

(b) Prove that if W is a we some then it is a subspace.

As W is nonempty, prck  $w_i \in W$ . By (ii), we have  $(-1)w_1 + w_1 = 0_V$  is in  $W_i$ . W is closed under addition as if  $w_1, w_2 \in W$  we can take a = 1 in (ii) to get

I'W,  $+W_2 = W_1 + W_2$ is in W: Finally, W is closed under scalar mult. since given  $W_1 \in W$  and  $\alpha \in \mathbb{R}$  we apply (ii) with  $W_2 = O_V$  to learn  $\alpha \cdot W_1 + O_V = \alpha W_1$  is in W. So W is a subspace as well as being awasome.

- **3.** Let *S* in  $Mat_{2\times 2}(\mathbb{R})$  be the subspace of symmetric matrices, i.e., those whose transpose  $A^t$  is equal to A.
  - (a) Give an explicit basis for S, carefully justifying your answer.

Claim: 
$$\beta = \left\{ A = \begin{pmatrix} 10 \\ 00 \end{pmatrix}, B = \begin{pmatrix} 00 \\ 01 \end{pmatrix}, C = \begin{pmatrix} 01 \\ 10 \end{pmatrix} \right\}$$
 is a basis for S.

As 
$$aA + bB + cC = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$
 and 0 in  $Mat_{2+2}(R)$  is  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 

this implies a=b=c=0. So B is linearly indep.

Span: Suppose 
$$M = \begin{pmatrix} u \vee \\ \times \gamma \end{pmatrix}$$
 is in S. As  $M^t = \begin{pmatrix} u \times \\ \vee \gamma \end{pmatrix} = M$  we have  $x = V$  and  $M = \begin{pmatrix} u \vee \\ \vee \gamma \end{pmatrix}$ . Then M is in span  $(\beta)$  with

As B is lin indep and spans S it is a basis for S.

(b) What is the dimension of *S* and why?

$$\dim S = \# \text{ any } = \#\beta = 3$$

**4.** Suppose  $\{w_1, w_2\}$  are linearly independent vectors in a vector space V. For v in V, prove that if  $\{v, w_1, w_2\}$  is linearly dependent then v is in  $\text{span}(w_1, w_2)$ .

Suppose aov+a,w,+azwz=0 with some a; ≠0.

If  $a_0 = 0$ , then  $a_1w_1 + a_2w_2 = 0$  which forces  $a_1 = a_2 = 0$  as  $\frac{1}{2}w_1, w_2\frac{1}{2}$  is linearly indep. But then all  $a_i = 0$ , contradicting our assumption. So must

have  $a_0 \neq 0$ . Then  $v = -\frac{a_1}{a_0} w_1 - \frac{a_2}{a_0} w_2$ 

and so  $v \in span(w_1, w_2)$  as required.

- **5.** Suppose V is a vector space where dim V=2.
  - (a) For any subspace  $W \subset V$ , what are the possibilities for dim W? (2 points)

As  $W \subseteq V$ , we know dim  $W \subseteq \dim V$ . As O is the minimum possible dimension for a vectors pace, we have 3 possibilities for dim  $W: \{0, 1, 2\}$ 

(b) Suppose  $W_1 \subset W_2 \subset W_3 \subset W_4$  are all subspaces of V. Prove that at least two of  $W_1, W_2, W_3$  and  $W_4$  are the same. Hint: Use part (a). **(5 points)** 

By (a), there are 3 possible values for each dim Wi. As there are 4 subspaces Wi, at least one value of dim Wi must repeat.

Thus, there exists i < j with dim Wi = dim Wj.

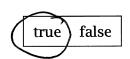
As  $W_i \subseteq W_j$ , the fact that they have the same dimension implies  $W_i = W_j$  as desired.

Note: The fact that  $W_i \subseteq W_j$  is crucial here. After all,  $\mathbb{R}^2$  contains infinitely many subspaces of dimension 1.

- (1 point each) 6. Circle true or false as appropriate; you do not need to provide any justification. (a) If  $A \in Mat_{3\times 3}(\mathbb{R})$  is row equivalent to a matrix in RREF that has no zero false true rows then the linear system  $\mathcal{LS}(A)$  is consistent. (b) If  $\{u, v, w\}$  is a basis for  $\mathbb{R}^3$  then  $\{u - v, v - w, w - u\}$  is also a basis false true for  $\mathbb{R}^3$ . (c) The set  $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$  is a subspace of  $\mathbb{R}^3$ . false (d) The elements  $f(t) = \sin^2(t)$ ,  $g(t) = \cos^2(t)$  and h(t) = 1 are linearly true independent in  $\mathcal{F}(\mathbb{R}, \mathbb{R})$ . false (e) Suppose a linear system with 4 variables and 6 equations has (1, 2, 0, 1)true and (3,0,1,5) as solutions. Then the total number of solutions to this
  - system is finite.

(f) The set  $\{a \sin(t) + b \cos(t) \mid a, b \in \mathbb{R}\}\$  is a subspace of  $\mathcal{F}(\mathbb{R}, \mathbb{R})$ .

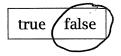
- false
- (g) With the terminology of Problem 2, if a subset W of a vector space V is awesome then it is a subspace.



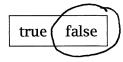
- 6. Circle true or false as appropriate; you do not need to provide any justification.
- false
- (a) If  $A \in \text{Mat}_{3\times 3}(\mathbb{R})$  is row equivalent to a matrix in RREF that has no zero rows then the linear system  $\mathcal{LS}(A)$  is inconsistent.

(1 point each)

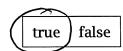
(b) If  $\{u, v, w\}$  is a basis for  $\mathbb{R}^3$  then  $\{u - v, v - w, w - u\}$  is also a basis for  $\mathbb{R}^3$ .



(c) The set  $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$  is a subspace of  $\mathbb{R}^3$ .



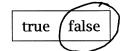
(d) The elements  $f(t) = \sin^2(t)$ ,  $g(t) = \cos^2(t)$  and h(t) = 1 are linearly dependent in  $\mathcal{F}(\mathbb{R}, \mathbb{R})$ .



(e) The set  $\{ae^t + be^{-t} \mid a, b \in \mathbb{R}\}$  is a subspace of  $\mathcal{F}(\mathbb{R}, \mathbb{R})$ .



(f) Suppose a linear system with 4 variables and 6 equations has (1, 2, 0, 1) and (3,0,1,5) as solutions. Then the total number of solutions to this system is finite.



(g) With the terminology of Problem 2, if a subset W of a vector space V is awesome then it is a subspace.

