1. Consider the matrix $A = \begin{pmatrix} 3 & 1 & -5 & 0 & 5 \\ 2 & 1 & -3 & 1 & 7 \\ 1 & 1 & -1 & 1 & 6 \end{pmatrix}$ which is row equivalent to $B = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$.

(a) What is the relationship between the solution sets to $\mathcal{LS}(A)$ and $\mathcal{LS}(B)$? **(1 point)**

As A and B are row equivalent, the two solution sets are the same

(b) Find a matrix C that is row equivalent to A and is in reduced row echelon form. Label any row operations you preform. **(2 points)**

$$A \xrightarrow[\text{ops}]{\text{row}} B \xrightarrow{-R_3+R_2} \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} = C$$

(c) Parameterize all solutions to the linear system $\mathcal{LS}(A)$. **(3 points)** This is the same as solving $\mathcal{LS}(C)$ which has one free variable corr. to the 3^{rd} column. Our eqns are

$$\begin{aligned} x_1 \quad - 2x_3 &= 1 \\ x_2 + x_3 &= 2 \\ x_4 &= 3 \end{aligned}$$

and hence our param. solutions are $\{(1+2t, 2-t, t, 3) \mid t \in \mathbb{R}$.

(d) Find the dimension of the rowspace $\mathcal{R}(A)$ of A. **(2 points)**

Know $\mathcal{R}(A) = \mathcal{R}(C)$. As C is in RREF, a basis for C is the set of all non zero rows. So $\dim \mathcal{R}(A) = 3$.

9am version

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(a) What is the relationship between the solution sets to $\mathcal{LS}(A)$ and $\mathcal{LS}(B)$? **(1 point)**

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and hence our param.

solutions are $\{(1+2t, 2-t, t, 3) \mid t \in \mathbb{R}\}$

(d) Find the dimension of the nullspace $\mathcal{N}(A)$ of A. **(2 points)**

Know $\mathcal{N}(A) = \mathcal{N}(C) =$ sol'ns to $\mathcal{LS}(C \,\vdots\, \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix})$.

As C is in RREF, $\dim \mathcal{N}(C) = \#$ of non pivot cols $= 2$

So $\dim \mathcal{N}(A) = 2$ as well.

10 am version

2. Let W be a subset of a vector space V. We say that W is *awesome* when both:

(i) The subset W is nonempty.

(ii) For all w_1 and w_2 in W and $a \in \mathbb{R}$, the linear combination $aw_1 + w_2$ is also in W.

(a) Prove that if W is a subspace then it is awesome.

As W is a subspace, it contains the 0 vector; in particular, it is nonempty and so satisfies (i).

As a subspace, W is closed under addition and scalar mult. So if $w_1, w_2 \in W$ and $a \in \mathbb{R}$ we have $aw_1 \in W$ and hence $aw_1 + w_2$ is in W. So W has (ii) as well and hence is awesome.

(b) Prove that if W is awesome then it is a subspace.

As W is nonempty, pick $w_1 \in W$. By (ii), we have $(-1)w_1 + w_1 = 0_V$ is in W. W is closed under addition as if $w_1, w_2 \in W$ we can take $a = 1$ in (ii) to get

$$1 \cdot w_1 + w_2 = w_1 + w_2$$

is in W. Finally, W is closed under scalar mult. since given $w_1 \in W$ and $a \in \mathbb{R}$ we apply (ii) with $w_2 = 0_V$ to learn $a \cdot w_1 + 0_V = aw_1$ is in W.

So W is a subspace as well as being awesome.

3. Let S in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ be the subspace of symmetric matrices, i.e., those whose transpose A^t is equal to A.

(a) Give an explicit basis for S, carefully justifying your answer.

<u>Claim</u>: $\beta = \left\{ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis for S.

<u>linear independence</u>: Suppose $aA + bB + cC = O$.
As $aA + bB + cC = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ and O in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
this implies $a = b = c = 0$. So β is linearly indep.

<u>Span</u>: Suppose $M = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$ is in S. As $M^t = \begin{pmatrix} u & x \\ v & y \end{pmatrix} = M$
we have $x = v$ and $M = \begin{pmatrix} u & v \\ v & y \end{pmatrix}$. Then M is
in span (β) with

$$M = uA + yB + vC$$

As β is lin indep and spans S it is a basis for S.

(b) What is the dimension of S and why?

$$\dim S = \# \underset{\text{basis}}{\text{any}} = \#\beta = 3$$

4. Suppose $\{w_1, w_2\}$ are linearly independent vectors in a vector space V. For v in V, prove that if $\{v, w_1, w_2\}$ is linearly dependent then v is in $\mathrm{span}(w_1, w_2)$.

Suppose $a_0 v + a_1 w_1 + a_2 w_2 = 0$ with some $a_i \neq 0$.
If $a_0 = 0$, then $a_1 w_1 + a_2 w_2 = 0$ which forces $a_1 = a_2 = 0$ as $\{w_1, w_2\}$ is linearly indep. But then all $a_i = 0$, contradicting our assumption. So must have $a_0 \neq 0$. Then $v = -\frac{a_1}{a_0} w_1 - \frac{a_2}{a_0} w_2$
and so $v \in \mathrm{span}(w_1, w_2)$ as required.

5. Suppose V is a vector space where $\dim V = 2$.

(a) For any subspace $W \subset V$, what are the possibilities for $\dim W$? **(2 points)**

As $W \subseteq V$, we know $\dim W \leq \dim V$.
As 0 is the minimum possible dimension for a vectorspace, we have 3 possibilities for $\dim W$: $\{0, 1, 2\}$

(b) Suppose $W_1 \subset W_2 \subset W_3 \subset W_4$ are all subspaces of V. Prove that at least two of W_1, W_2, W_3 and W_4 are the same. Hint: Use part (a). **(5 points)**

By (a), there are 3 possible values for each $\dim W_i$. As there are 4 subspaces W_i, at least one value of $\dim W_i$ must repeat.
Thus, there exists $i < j$ with $\dim W_i = \dim W_j$.
As $W_i \subseteq W_j$, the fact that they have the same dimension implies $W_i = W_j$ as desired.

Note: The fact that $W_i \subseteq W_j$ is crucial here. After all, $\mathbb{R}^2$ contains infinitely many subspaces of dimension 1.

6. Circle true or false as appropriate; you do **not** need to provide any justification. **(1 point each)**

(a) If $A \in \mathrm{Mat}_{3\times3}(\mathbb{R})$ is row equivalent to a matrix in RREF that has no zero rows then the linear system $\mathcal{LS}(A)$ is consistent.

true	false

(b) If $\{u, v, w\}$ is a basis for $\mathbb{R}^3$ then $\{u - v, v - w, w - u\}$ is also a basis for $\mathbb{R}^3$.

true	false

(c) The set $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$.

true	false

(d) The elements $f(t) = \sin^2(t)$, $g(t) = \cos^2(t)$ and $h(t) = 1$ are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

true	false

(e) Suppose a linear system with 4 variables and 6 equations has $(1, 2, 0, 1)$ and $(3, 0, 1, 5)$ as solutions. Then the total number of solutions to this system is finite.

true	false

(f) The set $\{a \sin(t) + b \cos(t) \mid a, b \in \mathbb{R}\}$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

true	false

(g) With the terminology of Problem 2, if a subset W of a vector space V is awesome then it is a subspace.

true	false

9am version.

6. Circle true or false as appropriate; you do **not** need to provide any justification. **(1 point each)**

(a) If $A \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ is row equivalent to a matrix in RREF that has no zero rows then the linear system $\mathcal{LS}(A)$ is inconsistent.

true	false

(b) If $\{u, v, w\}$ is a basis for $\mathbb{R}^3$ then $\{u - v, v - w, w - u\}$ is also a basis for $\mathbb{R}^3$.

true	false

(c) The set $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$ is a subspace of $\mathbb{R}^3$.

true	false

(d) The elements $f(t) = \sin^2(t)$, $g(t) = \cos^2(t)$ and $h(t) = 1$ are linearly dependent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

true	false

(e) The set $\{ae^t + be^{-t} \mid a, b \in \mathbb{R}\}$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

true	false

(f) Suppose a linear system with 4 variables and 6 equations has $(1, 2, 0, 1)$ and $(3, 0, 1, 5)$ as solutions. Then the total number of solutions to this system is finite.

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10 am version