Math 416: Midterm the First

Wednesday, Feburary 17, 2016

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There are 47 points possible on this exam. Take care to note that the problems are not weighted equally. Calculators, phones, books, notes (beyond the allowed one page) and suchlike aids to gracious living are not permitted. Show all your work as credit will not be given for correct answers without proper justification, except where indicated in Problems 5 and 6.

Do not start until instructed.

Good luck!

1. (a) Parameterize all solutions to the linear system $\mathcal{LS}(A)$ where $A = \begin{pmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 & 4 \end{pmatrix}$.

(4 points)

(b) Find a matrix C row equivalent to $B = \begin{pmatrix} -2 & 1 & -6 \\ 2 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$ that is in reduced row echelon form. Please label your individual row operations.

(4 points)

(c) Use the fact that $v = (3, 0, -1)$ is in $\mathcal{N}(B)$ to give a check for your answer for C in part (b).

(1 point)

2. Suppose $\{u, v\}$ is a basis for a vector space V over $\mathbb{R}$.

(a) What is the dimension of V? **(1 point)**

(b) If both a and b are nonzero real numbers, prove that $\beta = \{au, bv\}$ is also a basis of V. **(5 points)**

(c) Must $\beta' = \{u + v, u\}$ also be a basis for V? Prove or give a counterexample. **(4 points)**

3. Consider the subset W of $M_{2\times2}(\mathbb{R})$ consisting of matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a + d = 0$.

For example, $\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ is in W but $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ is not.

(a) Prove that W is a subspace. **(4 points)**

(b) Find basis for W. **(4 points)**

(c) What are the dimensions of $M_{2\times2}(\mathbb{R})$ and W? **(2 points)**

4. Suppose $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are linearly independent subsets of a vector space V over $\mathbb{R}$. Set $W = \text{span}(\{x_1, x_2, y_1, y_2\})$.

(a) Prove that $2 \le \dim(W) \le 4$. **(4 points)**

(b) Suppose that the only vector contained in both $X = \text{span}(\{x_1, x_2\})$ and $Y = \text{span}(\{y_1, y_2\})$ is 0. Prove that $\dim(W) = 4$. **(4 points)**

(c) Use part (b) to show that any two planes in $\mathbb{R}^3$ that contain the origin intersect in at least a line. **(2 points)**

5. Mark each statement as true or false. You do *not* need to justify your answers. **(1 point each)**

(a) The plane given by $x + 3y = 0$ is a subspace of $\mathbb{R}^3$.

True **False**

(b) If W_1 and W_2 are subspaces of $\mathbb{R}^2$ then their union $W_1 \cup W_2$ is also a subspace.

True **False**

(c) Suppose E is the subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ consisting of functions that are even, that is, $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ where $f(-t) = f(t)$ for all $t \in \mathbb{R}$. Then E is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

True **False**

(d) The vectors $\{(1, 2, 0), (0, 1, 0), (1, 0, -1)\}$ form a basis for $\mathbb{R}^3$.

True **False**

(e) The matrices $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ are row equivalent.

True **False**

6. For these last two problems, you do *not* need to justify your answers.

(a) Write the zero vector in the vector space $M_{3\times 2}(\mathbb{R})$. **(1 point)**

(b) Suppose $u_1 = (2, 3, 1)$, $u_2 = (3, 4, 1)$, and $u_3 = (2, -1, 1)$. Give a matrix A so that $\mathcal{LS}(A)$ has a solution if and only if $v = (1, 0, 1)$ is a linear combination of $\{u_1, u_2, u_3\}$. **(2 points)**

Extra Credit. Consider the set $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ with the following two operations:

- Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1,\ y_1 + y_2 + 1)$
- Scalar multiplication: $a(x, y) = (ax + a - 1,\ ay + a - 1)$.

Prove or disprove: With these operations, V is a vector space over $\mathbb{R}$. **(3 points)**