**1.** (a) Parameterize all solutions to the linear system $\mathcal{LS}(A)$ where $A = \begin{pmatrix} 1 & 0 & -3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$.

**(4 points)**

$$x_1 - 3x_3 = 1$$
$$x_2 + 2x_3 = 7$$
$$x_4 = 4$$

Free var $x_3 = t$

$$\text{Sol'ns} = \{(1+3t, 7-2t, t, 4) \mid t \in \mathbb{R}\}$$

(b) Find a matrix $C$ row equivalent to $B = \begin{pmatrix} 3 & 6 & 5 \\ 2 & 4 & 7 \\ 1 & 2 & -1 \end{pmatrix}$ that is in reduced row echelon form. Please label your individual row operations.

**(4 points)**

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ 3 & 6 & 5 \end{pmatrix} \xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 9 \\ 3 & 6 & 5 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{\frac{1}{9}R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 3 & 6 & 5 \end{pmatrix} \xrightarrow{-5R_2+R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 3 & 6 & 0 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{R_2+R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 6 & 0 \end{pmatrix} \xrightarrow{-3R_1+R_3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = C$$

(c) Use the fact that $v = (2, -1, 0)$ is in $\mathcal{N}(B)$ to give a check for your answer for $C$ in part (b).

**(1 point)** As $B$ and $C$ are row equiv, have $\mathcal{N}(B) = \mathcal{N}(C)$. So if b) is correct, $v = (2, -1, 0)$ will be in $\mathcal{N}(C) \Leftrightarrow \left\{ \begin{array}{l} x_1 + 2x_2 = 0 \\ \text{and } x_3 = 0 \end{array} \right\}$ which is the case.

1. (a) Parameterize all solutions to the linear system $\mathcal{LS}(A)$ where $A = \begin{pmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 & 4 \end{pmatrix}$.

**(4 points)**

Eqns are: $x_1 + 2x_4 = 3$ $\qquad x_4 = \text{free} = t$

$x_2 = 5$

$x_3 - x_4 = 4$

Sol'ns $= \{(3-2t, 5, 4+t, t) \mid t \in \mathbb{R}\}$

(b) Find a matrix $C$ row equivalent to $B = \begin{pmatrix} -2 & 1 & -6 \\ 2 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$ that is in reduced row echelon form. Please label your individual row operations.

**(4 points)**

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ -2 & 1 & -6 \end{pmatrix} \xrightarrow{-2R_1 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -2 & 1 & -6 \end{pmatrix} -$$

$$\xrightarrow{-2R_2 + R_1} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ -2 & 1 & -6 \end{pmatrix} \xrightarrow{-R_2 + R_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ -2 & 0 & -6 \end{pmatrix}$$

$$\xrightarrow{2R_1 + R_3} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = C$$

(c) Use the fact that $v = (3, 0, -1)$ is in $\mathcal{N}(B)$ to give a check for your answer for $C$ in part (b).

**(1 point)** As $B$ and $C$ are row equiv, they have the same nullsp. So we check that $v \in \mathcal{N}(C)$:

$\left\{ \begin{array}{l} x_1 + 3x_3 = 0 \\ x_2 = 0 \end{array} \right\}$ does indeed hold for $x_1 = 3$, $x_2 = 0$, $x_3 = -1$

2. Suppose $\{u, v\}$ is a basis for a vector space $V$ over $\mathbb{R}$.

(a) What is the dimension of $V$? **(1 point)**

$\dim V = \#\left(\begin{smallmatrix}\text{any}\\ \text{basis}\end{smallmatrix}\right) = 2$

(b) If both $a$ and $b$ are nonzero real numbers, prove that $\beta = \{au, bv\}$ is also a basis of $V$. **(5 points)**

As $\beta$ contains 2 vectors and $\dim V = 2$, it is enough to prove that $\mathrm{span}(\beta) = V$. Given $W \in V$, as $\{u, v\}$ is a basis there are $c, d \in \mathbb{R}$ with $W = cu + dv$. As neither $a$ or $b$ is $0$, we get to write

$W = \left(\frac{c}{a}\right)(au) + \left(\frac{d}{b}\right)(bv)$ and so $\mathrm{span}(\beta) = V$ as required.

~~Thus span(β)~~

(c) Must $\beta' = \{u + v, u\}$ also be a basis for $V$? Prove or give a counterexample. **(4 points)**

Yes it is a basis. As in b) need only check that $\mathrm{span}(\beta') = V$. Any $w \in V$ is

$w = cu + dv$

$= d(u+v) + (c-d)u$

we indeed have $w \in \mathrm{span}(\beta')$ as desired.

3. Consider the subset $W$ of $P_2(\mathbb{R})$ consisting of polynomials $f$ where $f(1) = 0$. For example, $x - 1$ and $x^2 + x - 2$ are in $W$ but $x^2 + x + 2$ is not. 

(a) Prove that $W$ is a subspace. **(4 points)**

Any $f \in P_2(\mathbb{R})$ is of the form $f(x) = ax^2 + bx + c$; as $f(1) = a + b + c$, we have $W = \{ax^2 + bx - (a+b) \mid a, b \in \mathbb{R}\}$. We now check the 3 conditions to be a subspace.

- Taking $a = b = 0$, we get the 0 polynomial. So $0 \in W$ as required.
- If $f, g \in W$, say $f = ax^2 + bx - (a+b)$ and $g = cx^2 + dx - (c+d)$ then $f + g = (a+c)x^2 + (b+d)x - (a+b+c+d)$ which is also in $W$.
- Finally, if $f \in W$ as before an $c \in \mathbb{R}$ we have $c \cdot f = (ac)x^2 + (bc)x - (ac+bc)$ which is also in $W$.

(b) Find basis for $W$. **(4 points)**

Claim: $\beta = \{x^2 - 1, x - 1\}$ is a basis.

Span: Any $f$ in $W$ is $ax^2 + bx - (a+b) = a(x^2 - 1) + b(x - 1)$
$\Rightarrow f \in \operatorname{span}(\beta)$

Lin. Indep: If $a(x^2 - 1) + b(x - 1) = 0$ then $a = 0$ & $b = 0$ since the quadratic term is $ax^2$ and the linear term is $bx$.

(c) What are the dimensions of $P_2(\mathbb{R})$ and $W$? **(2 points)**

$\dim P_2(\mathbb{R}) = 3$ $\quad\quad$ $\dim W = \# \beta = 2$.

3. Consider the subset $W$ of $M_{2\times 2}(\mathbb{R})$ consisting of matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a + d = 0$.

For example, $\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ is in $W$ but $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ is not.

(a) Prove that $W$ is a subspace. **(4 points)**

Have $W = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \ a, b, c \in \mathbb{R} \right\}$

Check 3 conditions:

- Have $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ so $W$ does contain the zero in $M_{2\times2}(\mathbb{R})$.
- Closed under addition:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & -(a_1 + a_2) \end{pmatrix}$$

which is indeed in $W$.

- Closed under scalar mult.

$$d \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} da & db \\ dc & -da \end{pmatrix}$$ which is in $W$.

So $W$ is a subspace of $M_{2\times2}(\mathbb{R})$.

(b) Find basis for $W$. **(4 points)**

Claim: $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis for $W$.

Span: Given elt of $W$ have $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

so $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \text{span}(\beta)$.

Linear indep: If $u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + v \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + w \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ w & -u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

get $u = v = w = 0$. So $\beta$ is lin. indep.

(c) What are the dimensions of $M_{2\times 2}(\mathbb{R})$ and $W$? **(2 points)**

$\dim M_{2\times2}(\mathbb{R}) = 4$

$\dim W = \#\beta = 3$.

4. Suppose $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are linearly independent subsets of a vector space $V$ over $\mathbb{R}$. Set $W = \text{span}(\{x_1, x_2, y_1, y_2\})$. Ex:1

(a) Prove that $2 \le \dim(W) \le 4$. **(4 points)** As $W$ is spanned by a set of four vectors, it must have $\dim \le 4$.

As $W$ contains a linearly indep subset of size 2, namely $\{x_1, x_2\}$, we also have $\dim W \geq 2$.

So $2 \le \dim W \le 4$ as required.

(b) Suppose that the only vector contained in both $X = \text{span}(\{x_1, x_2\})$ and $Y = \text{span}(\{x_1, x_2\})$ is 0. Prove that $\dim(W) = 4$. **(4 points)**

It suffices to prove that $\beta = \{x_1, x_2, y_1, y_2\}$ is a basis for $W$ as $\#\beta = 4$. Since $W = \text{span}(\beta)$ by definition, we really only need to check linear independence.

Suppose we have scalars with

$$a_1 x_1 + a_2 x_2 + b_1 y_1 + b_2 y_2 = 0$$

Then $a_1 x_1 + a_2 x_2 = -b_1 y_1 - b_2 y_2$ is in both $X$ and $Y$ and hence is 0. As $a_1 x_1 + a_2 x_2 = 0$ and $\{x_1, x_2\}$ is linearly indep, we get $a_1 = a_2 = 0$. Similarly $b_1 y_1 + b_2 y_2 = 0$ and $\{y_1, y_2\}$ are lin. indep forces $b_1 = b_2 = 0$. So $\beta$ is lin. indep.

(c) Use part (b) to show that any two planes in $\mathbb{R}^3$ that contain the origin intersect in at least a line. **(2 points)**

Choose $x_1, x_2, y_2, y_2 \in \mathbb{R}^3$ so that $X$ and $Y$ as above are the two planes in question. If $X \cap Y = \{0\}$, by (b) we have $\dim W = 4$ which is impossible since $W \subseteq \mathbb{R}^3$ which has dim 3. So $X \cap Y$ contains a nonzero vector $v$, and so $\text{span}\{v\}$ is the line we seek.

5. Mark each statement as true or false. You do *not* need to justify your answers. **(1 point each)**

(a) The plane given by $x + 2y + 3z = 4$ is a subspace of $\mathbb{R}^3$.

**True** **False**

(b) Suppose $E$ is the subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ consisting of functions that are even, that is, $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ where $f(-t) = t$ for all $t \in \mathbb{R}$. Then $E$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

**True** **False**

(c) If $W_1$ and $W_2$ are subspaces of $\mathbb{R}^2$ then their union $W_1 \cup W_2$ is also a subspace.

**True** **False**

(d) The matrices $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ are row equivalent.

**True** **False**

(e) The vectors $\{(1,0,0), (0,1,0), (1,0,-1)\}$ form a basis for $\mathbb{R}^3$.

**True** **False**

6. For these last two problems, you do *not* need to justify your answers.

(a) Write the zero vector in the vector space $M_{2\times 3}(\mathbb{R})$. **(1 point)**

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Suppose $u_1 = (2,3,1)$, $u_2 = (3,4,1)$, and $u_3 = (2,-1,1)$. Write down a matrix $A$ where $\{u_1, u_2, u_3\}$ are linearly independent if and only if $\mathcal{N}(A)$ contains a nonzero vector. **(2 points)**

$$\begin{pmatrix} 2 & 3 & 2 \\ 3 & 4 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

5. Mark each statement as true or false. You do *not* need to justify your answers. **(1 point each)**

(a) The plane given by $x + 3y = 0$ is a subspace of $\mathbb{R}^3$.

True False

(b) If $W_1$ and $W_2$ are subspaces of $\mathbb{R}^2$ then their union $W_1 \cup W_2$ is also a subspace.

True False

(c) Suppose $E$ is the subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ consisting of functions that are even, that is, $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ where $f(-t) = t$ for all $t \in \mathbb{R}$. Then $E$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

True False

(d) The vectors $\{(1,2,0), (0,1,0), (1,0,-1)\}$ form a basis for $\mathbb{R}^3$.

True False

(e) The matrices $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ are row equivalent.

True False

6. For these last two problems, you do *not* need to justify your answers.

(a) Write the zero vector in the vector space $M_{3\times 2}(\mathbb{R})$. **(1 point)**

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(b) Suppose $u_1 = (2,3,1)$, $u_2 = (3,4,1)$, and $u_3 = (2,-1,1)$. Give a matrix $A$ so that $\mathcal{LS}(A)$ has a solution if and only if $v = (1,0,1)$ is a linear combination of $\{u_1, u_2, u_3\}$. **(2 points)**

$$\begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 4 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



Page score: