1. See back of book.

2. Suppose $M \in M_{n\times n}(\mathbb{R})$ and $v \in \mathbb{R}^n$ with all entries $\geq 0$. Set $u = (1, \dots, 1) \in \mathbb{R}^n$.

a) $M$ is transition $\Leftrightarrow M^t u = u$.

Pf: Note that $M^t u$ is the vector whose $i^{th}$ entry is the sum of the $i^{th}$ row of $M^t$. Thus

$$M^t u = u \Leftrightarrow \text{every row of } M^t \text{ sums to } 1$$
$$\Leftrightarrow \text{every column of } M \text{ sums to } 1,$$
i.e. $M$ is transition.

b) $v$ is a probability vector $\Leftrightarrow u^t v = (1)$

Pf: Note that $u^t v$ is the $1\times 1$ matrix whose entry is the sum of entries of $v$. Thus

$u^t v = (1) \Leftrightarrow \sum v_i = 1 \Leftrightarrow v$ is a probability vector.

Cor a) If $A, B \in M_{n\times n}(\mathbb{R})$ are transition, so is $AB$

Pf: We apply part (a) above: $(AB)^t u = B^t A^t u$

$$= B^t(\underbrace{A^t u}_{\text{as } A^t \text{ is transition}}) = \underbrace{B^t u = u}_{\text{as } B^t \text{ is transition}}$$

So $(AB)^t u = u$ and hence $AB$ is transition by part (a).

2. Continued

<u>Cor b)</u> If $A \in M_{n\times n}(\mathbb{R})$ is a transition matrix and $v \in \mathbb{R}^n$ is a probability vector, then $Av$ is a probability vector.

<u>Pf</u>: We have $u^t(Av) = (u^tA)v = (A^tu)^t v$

$$\underset{\substack{\uparrow \\ \text{by part (a)}}}{=} u^t v \underset{\substack{\uparrow \\ \text{by part (b)}}}{=} (1)$$ and hence $Av$ is a prob. vector.

3. Suppose the subspace $W \subseteq \mathbb{R}^n$ contains a nonzero vector $w$ with all $w_i \geq 0$. Then if $s = \sum_{i=1}^{n} w_i$, the vector $\frac{1}{s}w$ is a probability vector in $W$.

If $v_1$ and $v_2$ are two probability vectors in $W$, then as $\dim W = 1$ we have $v_1 = c\,v_2$ for some $c \in \mathbb{R}$.

Summing the entries of $v_1$ and $c\,v_2$ shows that we must have $c = 1$ as needed. $\square$

4. Our transition matrix here is

$$\begin{array}{ll} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{ll} \text{Win} & 1 \\ \text{Start} & 2 \\ & 3 \\ \text{Lose} & 4 \end{array} & \begin{pmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 0 \\ 0 & 0 & 2/3 & 1 \end{pmatrix} \end{array}$$

4. Continued

which has char poly $(1-t)^2 \det\begin{pmatrix} -t & 1/3 \\ 2/3 & -t \end{pmatrix}$

$$= (1-t)^2\left(t^2 - 2/9\right) = (1-t)^2\left(\frac{\sqrt{2}}{3} - t\right)\left(-\frac{\sqrt{2}}{3} - t\right)$$

and eigenspaces

$$E_1 = \text{span}\left(\{e_1, e_4\}\right)$$

$$E_{\sqrt{2}/3} = \text{span}\left(v_1 = (1, -3+\sqrt{2}, 2-3\sqrt{2}, 2\sqrt{2})\right)$$

$$E_{-\sqrt{2}/3} = \text{span}\left(v_2 = (1, -3-\sqrt{2}, 2+3\sqrt{2}, -2\sqrt{2})\right)$$

Our initial probability vector is

$$p_0 = e_2 = \frac{3}{7}e_1 + \frac{4}{7}e_2 - \frac{1}{14}(3+\sqrt{2})v_1 + \frac{1}{14}(-3+\sqrt{2})v_2$$

Hence inductively we have

$$A^n p_0 = 1^n \cdot \frac{3}{7}e_1 + 1^n \cdot \frac{4}{7}e_2 + \left(\frac{\sqrt{2}}{3}\right)^n\left(-\frac{1}{14}(3+\sqrt{2})\right)v_1 + \left(-\frac{\sqrt{2}}{3}\right)^n\left(\frac{1}{14}(-3+\sqrt{2})\right)v_2$$

and so as $n \to \infty$ we get

$$\lim_{n\to\infty} A^n p_0 = \frac{3}{7}e_1 + \frac{4}{7}e_2$$ and so prob. of winning is $3/7$.

5. See back of book.

6. a) Doesn't satisfy axiom (d) as $\langle (0,1), (0,1) \rangle$
$= 0 \cdot 0 - 1 \cdot 1 = -1$ which is negative.

b) Doesn't satisfy axiom (b) as
$$\langle 0 \cdot I_2, I_2 \rangle = tr(0 \cdot I_2 + I_2) = tr(I_2) = 2$$
which is not $0 \cdot \langle I_2, I_2 \rangle = 0 \cdot 4 = 0$.

~~8.~~

7. Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for a finite dim'l inner product space V.

a) If $\langle x, z \rangle = 0$ for all $z \in \beta$ then $x = 0$

<u>Proof</u>: We have $x = \sum_{i=1}^{n} a_i v_i$. Then
$$\langle x, x \rangle = \langle x, \sum a_i v_i \rangle = \sum a_i \overbrace{\langle x, v_i \rangle}^{\text{all } 0}$$
$$= \sum a_i \cdot 0 = 0$$
As $\|x\| = 0$ we must have $x = 0$.

7. Continued

b) If $\langle x,z\rangle = \langle y,z\rangle$ for all $z\in\beta$ then $x=y$.

Pf: As $\langle x-y,z\rangle = \langle x,z\rangle - \langle y,z\rangle = 0$ for all $z\in\beta$, by part (a) we have $x-y=0$, that is $x=y$.

8. Suppose $x,y\in V$ are orthogonal. Then $\|x+y\|^2 = \|x\|^2+\|y\|^2$

Pf: $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \overset{=0}{\langle x,y\rangle} + \underset{=0}{\langle y,x\rangle} + \langle y,y\rangle$

$= \|x\|^2+\|y\|^2$ $\blacksquare$

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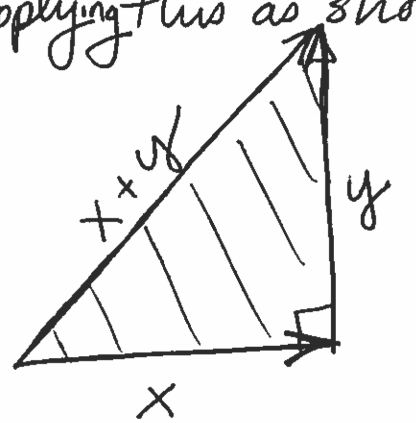
Applying this as shown to a Euclidean right triangle




gives the usual Pythagorean theorem.

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Now suppose $x,y\in V$. We must show $|\langle x,y\rangle| = \|x\|\|y\|$ if and only if one of $x$ and $y$ is a scalar multiple of the other.

Proof: ($\Leftarrow$) Suppose $y = cx$ (the case $x = cy$ is similar).

Then $|\langle x, y\rangle| = |\bar{c}\langle x, x\rangle| = |c|\,\|x\|^2 = \|x\| \cdot (|c|\,\|x\|)$

$= \|x\|\,\|y\|$ as needed.

($\Rightarrow$) Assume $y \neq 0$ as otherwise $y$ is a scalar mult of $x$. Set $a = \dfrac{\langle x, y\rangle}{\|y\|^2}$ and consider

$z = x - ay$. Then $\langle z, y\rangle = \langle x - ay, y\rangle$

$$= \langle x, y\rangle - a\langle y, y\rangle = \langle x, y\rangle - \frac{\langle x, y\rangle}{\|y\|^2}\|y\|^2 = 0.$$

As $y$ and $z$ are orthogonal, by the first part of this problem we have

$$\|x\|^2 = \|ay + z\|^2 = \|ay\|^2 + \|z\|^2 = |a|^2\|y\|^2 + \|z\|^2$$

$$= \frac{|\langle x, y\rangle|^2}{\|y\|^2} + \|z^2\| = \|x\|^2 + \|z\|^2$$

since we've assumed that $|\langle x, y\rangle| = \|x\|\,\|y\|$. Thus $\|z\|^2 = 0$ and so $z = 0$, i.e. $x = ay$ as needed. $\blacksquare$