1. Solution:

It's enough to discuss the following three operations:

(1). Interchanging any two rows. i, j.

Then, we have $I_n \xrightarrow{R} E = \begin{array}{c c} & \begin{array}{cccc} & i & j & \end{array} \\ \begin{array}{c} \\ i \\ j \\ \\ \end{array} & \begin{pmatrix} 1 & \cdots 0 & \cdots 0 & \cdots 0 \\ 0 & \cdots 0 & \cdots 1 & \cdots 0 \\ 0 & \cdots 1 & \cdots 0 & \cdots 0 \\ 0 & \cdots 0 & \cdots 0 & \cdots 1 \end{pmatrix} \end{array}$

Thus, $EA = \begin{pmatrix} 1 & \cdots 0 & \cdots 0 & \cdots 0 \\ 0 & \cdots 0 & \cdots 1 & \cdots 0 \\ 0 & \cdots 1 & \cdots 0 & \cdots 0 \\ 0 & \cdots 0 & \cdots 0 & \cdots 1 \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{j1} & \cdots & a_{jn} \\ a_{i1} & \cdots & a_{in} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

Therefore, $A \xrightarrow{R} EA$.

(2). multiplying row i with a.

Then, we have $I_n \xrightarrow{R} E = \begin{array}{c c} \begin{array}{c} \\ i \\ \\ \end{array} & \begin{pmatrix} 1 & \cdots & 0 \\ 0 \cdots & a & \cdots 0 \\ 0 & \cdots & 1 \end{pmatrix} \end{array}$

Thus, $EA = \begin{array}{c c} \begin{array}{c} \\ i \\ \\ \end{array} & \begin{pmatrix} 1 & \cdots 0 & \cdots 0 \\ 0 & \cdots a & \cdots 0 \\ 0 & \cdots 0 & \cdots 1 \end{pmatrix} \end{array} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a a_{i1} & \cdots & a a_{in} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

Therefore, $A \xrightarrow{R} EA$.

(3). adding $b \times$ row i to row j.

Then we have $I_n \xrightarrow{R} E = \begin{array}{c c} & \begin{array}{cccc} & i & j & \end{array} \\ \begin{array}{c} \\ i \\ j \\ \\ \end{array} & \begin{pmatrix} 1 & \cdots 0 & \cdots 0 & \cdots 0 \\ 0 & \cdots 1 & \cdots 0 & \cdots 0 \\ 0 & \cdots b & \cdots 1 & \cdots 0 \\ 0 & \cdots 0 & \cdots 0 & \cdots 0 \end{pmatrix} \end{array}$

Thus, $EA = \begin{pmatrix} 1 & \cdots 0 & \cdots 0 & \cdots 0 \\ 0 & \cdots 1 & \cdots 0 & \cdots 0 \\ 0 & \cdots b & \cdots 1 & \cdots 0 \\ 0 & \cdots 0 & \cdots 0 & \cdots 1 \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ b a_{i1} + a_{j1} & \cdots & b a_{in} + a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

Therefore, we have $A \xrightarrow{R} EA$.

$\square$.

2. Solution:

Since A, B are similar, there exists an invertible matrice $Q \in M_{n\times n}(\mathbb{R})$ such that $B = Q^{-1} A Q$.

Therefore, we have $\det(B) = \det(Q^{-1} A Q)$

$$= \det(Q^{-1}) \det(A) \det(Q) = [\det(Q)]^{-1} \det(A) \det(Q)$$

$$= \det(A).$$

$\square$.

3. Solution:

(a). Since Q is orthogonal, $QQ^t = I_n$.

Then we obtain $1 = \det(I_n) = \det(Q)\det(Q^t) = [\det(Q)]^2$

Therefore, $\det(Q) = \pm 1$.

(b). $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $Q^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q \cdot Q^T = I_2$ and $\det(Q) = 1$

$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $Q^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Q \cdot Q^T = I_2$ and $\det(Q) = -1$.

4. Solution:

(a) Suppose A is not invertible, then $\det(A) = 0$.

Therefore, we obtain $1 = \det(I_n) = \det(AB) = \det A \cdot \det B = 0$, which is a contradiction.

(b). Since A is invertible. ~~we have~~ there exists A^{-1} such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n.$$

Therefore, we obtain $B = \cancel{B}\, I_n \cdot B = (A^{-1} \cdot A)B = A^{-1}(AB) = A^{-1} I_n = A^{-1}$.

6. Solution:

a) "$\Rightarrow$" If T is invertible, let $A = [T]_\beta$, then $\det(A) \neq 0$.

Therefore $\det(A - 0 \cdot I_n) \neq 0$, which implies 0 is not an eigenvalue of T.

"$\Leftarrow$" If 0 is not an eigenvalue of T, then $\det(A - 0 \cdot I_n) \neq 0$, i.e. $\det(A) \neq 0$

Therefore, A is invertible.

b). ~~"$\Rightarrow$" Since λ^{-1} is an eigenvalue of T^{-1}, we have $\det(A^{-1} - \lambda^{-1} \cdot I_n) \neq 0$~~

"$\Leftarrow$" Since λ is an eigenvalue of T, there exists a nonzero vector v such that $T(v) = \lambda v$.

Therefore, we have $v = T^{-1}(T(v)) = T^{-1}(\lambda v) = \lambda T^{-1}(v)$

i.e. $T^{-1}(v) = \lambda^{-1} v$. Thus, λ^{-1} is an eigenvalue of T^{-1}

"$\Rightarrow$" Similarly as "$\Leftarrow$".

$\square$

7. Solution:

By condition, we have $T(v) = \lambda v$.

Thus, $T^m(v) = T(T^{m-1}(v)) = \cdots = T(T(\cdots(T(v))\cdots)) = T(T(\cdots(\lambda v)\cdots))$

$= \lambda^m v$.

$\square$.

Q5. (a) $[T]_\beta = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$

(c) $[T]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Q8. Compute eigenvalue

$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow \lambda_1 = -1$ or $\lambda_2 = 4$

Compute eigenvector

for $\lambda_1 = -1$ $(A + I)\, v = 0$

$\Rightarrow v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

for $\lambda_2 = 4$ $(A - 4I)\, w = 0$

$\Rightarrow w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Basis $\beta = \{v, w\}$

$Q = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ $D = Q^{-1} A Q = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

Q9. (b) $\lambda = -1, 1, 2$

$v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$$[T]_\beta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(h) $\lambda = -1, 1 \leftarrow$ multiplicity 3.

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

$$[T]_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$