1. As T being an isomorphism is the same saying T is invertible, the $(\Rightarrow)$ direction is page 3 of the class notes for Lecture 17.

For $(\Leftarrow)$ suppose γ is a basis for W. As $\#\beta = \#\gamma$, we get $\dim(V) = \dim(W)$ so it suffices to prove that T is onto to show that it is an isomorphism. Given $w \in W$ as γ is a basis there are scalars with

$$w = a_1 w_1 + \cdots + a_n w_n$$

Set $v = a_1 v_1 + \cdots + a_n v_n$ and note

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n)$$

$$= a_1 w_1 + \cdots + a w_n = w.$$

So we have found $v \in V$ with $T(v) = w$ as required to show T is onto. $\blacksquare$

2. See solutions in back of textbook.

3. Here are the parts not in the back of the book.

FIS §2.5

2(b): $\beta = \{(-1,3), (2,-1)\}$ and $\beta' = \{(0,10), (5,0)\}$

Set $\gamma = \{e_1, e_2\}$ and note

$$[I]_\beta^\gamma = \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix} \quad \text{and} \quad [I]_{\beta'}^\gamma = \begin{pmatrix} 0 & 5 \\ 10 & 0 \end{pmatrix}.$$

Thus

$$[I]_{\beta'}^{\beta} = [I]_\gamma^\beta \, [I]_{\beta'}^\gamma = \left([I]_\beta^\gamma\right)^{-1} [I]_\beta^\gamma$$

$$= \left(-\frac{1}{5}\begin{pmatrix} -1 & -2 \\ -3 & -1 \end{pmatrix}\right)\begin{pmatrix} 0 & 5 \\ 10 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

FIS §2.5 2(d): Same method, answer is $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$.

FIS §2.5 3(d): Again, use the intermediate basis $\gamma = \{1, x, x^2\}$ so that

$$[I]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad [I]_{\beta'}^\gamma = \begin{pmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$$

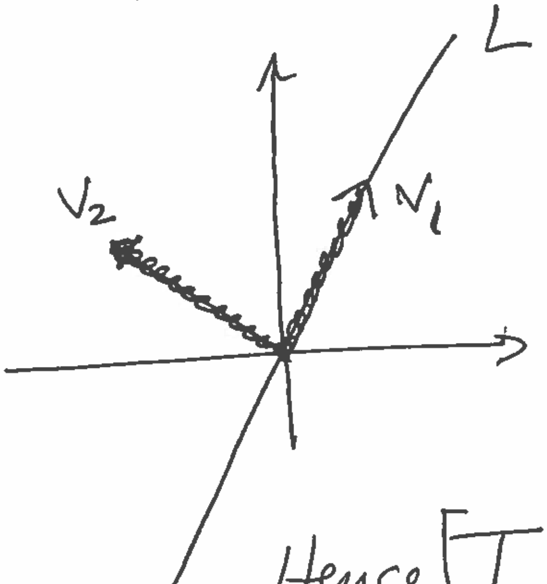
Thus

$$[I]_{\beta'}^{\beta} = \left([I]_{\beta}^{\gamma}\right)^{-1}[I]_{\beta'}^{\gamma} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$$

4. See back of book.

5. Let L be the line $y = mx$ with $m \neq 0$. Consider the basis $\beta' = \{v_1 = (1, m)$ and $v_2 = (-m, 1)\}$

Note here that v_1 is on L and that v_2 is perpendicular to L as $v_1 \cdot v_2 = 0$.



For (a), note that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection then $T(v_1) = v_1$ and $T(v_2) = -v_2$.

Hence $[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. To convert this to $\beta = \{e_1, e_2\}$, we have

$$[T]_{\beta} = [I]_{\beta'}^{\beta}[T]_{\beta'}[I]_{\beta}^{\beta'}$$

where $[I]^{\beta}_{\beta'} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$ and $[I]^{\beta'}_{\beta} = \left([I]^{\beta}_{\beta'}\right)^{-1}$

$= \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$. Thus multiplying out we get

$[T]_{\beta} = \frac{1}{1+m^2}\begin{pmatrix} -m^2+1 & +2m \\ +2m & m^2-1 \end{pmatrix}$ and so

$T(x,y) = [T_{\beta}]\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+m^2}\left((-m^2+1)x+2my,\ +2mx+(m^2-1)y\right)$

For part (b) have the same except with the projection one has $T(v_2)=0$ and so $[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence $[T]_{\beta} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$

and $T(x,y) = [T_{\beta}]\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+m^2}(x+my,\ mx+m^2y)$.

6. a) -2 b) 10 c) 0

7. a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and so

$\det(A^t) = ad - bc = \det(A)$.

b) So $B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ and $\det(B) = c \cdot b - ad$

$$= -(ad - bc) = -\det(A).$$

c) Also changes by a sign: If $C = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$, then

$\det(C) = bc - da = -(ad - bc) = -\det(A)$

d) Set $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. Then

$$\det(A) \cdot \det(B) = (A_{11}A_{22} - A_{12}A_{21}) \cdot (B_{11}B_{22} - B_{12}B_{21})$$
$$= A_{11}A_{22}B_{11}B_{22} - A_{11}A_{22}B_{12}B_{21} - A_{12}A_{22}B_{11}B_{22}$$
$$+ A_{12}A_{21}B_{12}B_{21}$$

and also

$$AB = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

So

$$\det(AB) = (A_{11}B_{11} + A_{12}B_{21}) \cdot (A_{21}B_{12} + A_{22}B_{22})$$
$$- (A_{21}B_{11} + A_{22}B_{21}) \cdot (A_{11}B_{12} + A_{12}B_{22})$$

which does indeed expand to our previous formula for $\det(A) \cdot \det(B)$.

8. FIS §4.1 # 10.

Set $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$

a) $$AC = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix} = (\det(A)) I_2$$

with the ~~se~~ case of CA similar.

b) Have $\det(C) = A_{11}A_{22} - (-A_{21})(-A_{12})$
$$= A_{11}A_{22} - A_{12}A_{21} = \det(A).$$

c) The classical adjoint of $A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$ is

$\begin{pmatrix} A_{11} & -A_{21} \\ -A_{12} & A_{22} \end{pmatrix}$ which is indeed C^t.

d) If A is invertible, we have $\det(A) \neq 0$, so it makes sense to consider $B = \frac{1}{\det(A)} C$.

By part a) we have
$$AB = A \cdot \left(\frac{1}{\det(A)} C\right) = \frac{1}{\det(A)} AC = I_2$$

and
$$BA = \left(\frac{1}{\det(A)} C\right) \cdot A = \frac{1}{\det(A)} (CA) = I_2.$$

So $A^{-1} = B$ as needed.

9. + 10. In back of book.