#2: §2.1 #2 : $T: \mathbb{R}^3 \to \mathbb{R}^2$ $T(a_1, a_2, a_3) = (a_1 - a_2, a_3)$

i) Suppose $v_1 = (a_1, a_2, a_3)$ and $v_2 = (b_1, b_2, b_3)$ are in $\mathbb{R}^3$ and $c \in \mathbb{R}$. Then

$$T(cv_1 + v_2) = T(ca_1 + b_1, ca_2 + b_2, ca_3 + b_3)$$
$$= (ca_1 + b_1 - (ca_2 + b_2), ca_3 + b_3)$$
$$= c(a_1 - a_2, a_3) + (b_1 - b_2, b_3) = cT(v_1) + T(v_2).$$

and so T is linear.

ii) $\mathcal{N}(T) = \{v \in \mathbb{R}^3 \mid T(v) = 0\} = \{(a, a, 0)\}$ which has basis $\beta = \{(1,1,0)\}$.

$R(T)$ contains $T(1,0,0) = (1,0)$ and $T(0,0,1) = (0,1)$. As $R(T)$ is a subspace, it must be $\mathbb{R}^2$ which has basis $\gamma = \{(1,0), (0,1)\}$.

iii) $\dim \mathcal{N}(T) + \dim R(T) = \#\beta + \#\gamma = 1 + 2 = 3$ which indeed is $\dim \mathbb{R}^3$ as promised by the Dimension Thm.

§2.1 #3: Similiar, with $\mathcal{N}(T) = \{0\}$ having basis $\beta = \emptyset$ and $R(T)$ having basis $\gamma = \{(1,0,0), (0,0,1)\}$.

#3: §2.1 #9

a) Not linear since $T(0,0) + T(0,0) = (1,0) + (1,0)$ $= (2,0)$ which is not $T((0,0)+(0,0)) =$ $T(0,0)) = (1,0)$.

b) Not linear since $T(2\cdot(0,1)) = T(0,2) = (0,4)$ but $2\cdot T(0,1) = 2\cdot(0,1) = (0,2)$.

c) Not linear since $T(2\cdot(\pi/2, 0)) = T(\pi, 0)$ $= (\sin(\pi), 0) = (0,0)$ but $2\cdot T(\pi/2, 0)$ $= 2\cdot(\sin \pi/2, 0) = (2,0)$.

#4: §2.1 #10 Set $v = (1,0)$ and $w = (1,1)$.

Then $T(2,3) = T(-v + 3w) = -T(v) + 3T(w)$ as T is linear

$= -(1,4) + 3\cdot(2,5) = (5,11)$.

As $\{T(v), T(w)\}$ are linearly indep, know $\dim R(T) \geq 2$ which means $\dim R(T) = 2$ as $R(T) \subseteq \mathbb{R}^2$. Now the Dimension Thm gives

$$\dim N(T) = \dim \mathbb{R}^2 - \dim R(T) = 2 - 2 = 0$$

and so $N(T) = \{0\}$. So T is 1-1 by Thm 2.4 of [FIS].

#5. <u>§2.1 #18</u>: For example, take $T(x,y) = (y,0)$ in which case $R(T) = \mathcal{N}(T) = x$-axis.

One way to find such an example is to note that if $R(T) = \mathcal{N}(T)$ then the Dim Thm gives

$$2\dim(R(T)) = \dim \mathbb{R}^2$$

which implies $\dim(R(T)) = 1$. So $R(T)$ has to be a line through 0 in $\mathbb{R}^2$; let's say the x-axis. For $R(T)$ to be the x-axis, must have $T(x,y) = (ax+by, 0)$ for some $a, b \in \mathbb{R}$. For such a T to have $\mathcal{N}(T) = x$-axis, must have

$$T(x,0) = (a\cdot x + b\cdot 0,\ 0) = (ax, 0) = 0$$

which suggests $a=0$. Taking $b=1$ gives the initial example.

HW4 Solution

3. [FIS ; Problem 9. a, b, c]

(a). Since $T(a_1+a_2, b_1+b_2) = (1, b_1+b_2) \neq (1, b_1) + (1, b_2) = T(a_1, b_1) + T(a_2, b_2)$,
T is NOT Linear.

(b) Since $T(2a_1, 2a_2) = (2a_1, 4a_1^2) \neq 2(a_1, a_1^2) = 2 \cdot T(a_1, a_2)$ for $|a_1| > 0$,
T is NOT linear.

(c) Since $T(2a_1, 2a_2) = (\sin 2a_1, 0) \neq 2(\sin a_1, 0) = 2T(a_1, a_2)$, for $\cos a_1 \neq 1$.
T is NOT linear.

5. We define a function $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (0, x)$
T is linear since $T(c_1x_1+c_2x_2, c_1y_1+c_2y_2) = (0, c_1x_1+c_2x_2)$
$= c_1(0, x_1) + c_2(0, x_2) = c_1T(x_1, y_1) + c_2T(x_2, y_2)$
Moreover, $N(T) = \{(0, x) \mid x \in \mathbb{R}\}$ and $R(T) = \{(0, x) \mid x \in \mathbb{R}\}$

6. (a) Suppose there is a one-to-one lin. Tran. T.
Since $\dim(V) = n$, we can find $v_1, \cdots, v_n$ such.
$c_1v_1 + \cdots + c_nv_n = 0 \Leftrightarrow c_1 = \cdots = c_n = 0$. $(*)$
$\therefore$ By our assumption, $T(v_1), \cdots, T(v_n)$ are all different,
moreover, $c_1T(v_1) + \cdots + c_nT(v_n) = 0 \Leftrightarrow c_1 = \cdots = c_n = 0$
[Otherwise, $\exists\, c_1, \cdots, c_n$ with $c_i \neq 0$ for some i, s.t.
$c_1T(v_1) + \cdots + c_nT(v_n) = T(c_1v_1 + \cdots + c_nv_n) = 0 = T(0)$.
Since T is one-to-one, $c_1v_1 + \cdots + c_nv_n = 0$, which is contradicted with $(*)$].
Since $\{T(v_1), \cdots, T(v_n)\} \subset W$ $\therefore$ $\dim W \geq n > m$. contradiction.

b) Suppose there exists an onto lin. trans. T.
Since $\dim V = n$. take $v_1, \cdots, v_n \in V$ s.t.

$$c_1 v_1 + \cdots + c_n v_n = 0 \iff c_1 = \cdots = c_n = 0. \quad (*)$$

By our assumption, there exist $w_1, \cdots, w_n$ s.t $T(w_i) = v_i$.
Moreover, w_i are all different. Otherwise, if $w_i = w_j$.
Then we have $w_i - w_j = 0 \Rightarrow T(w_i) - T(w_j) = 0$ i.e. $v_i - v_j = 0$,
which is contradicted with $(*)$
Besides, $w_1, \cdots, w_n$ are independent since

$$c_1 w_1 + \cdots + c_1 w_n = 0 \Rightarrow T(c_1 w_1 + \cdots + c_n w_n) = 0$$
$$\Rightarrow c_1 v_1 + \cdots + c_n v_n = 0 \Rightarrow c_1 = \cdots = c_n = 0$$

Since $\{w_1, \cdots, w_n\} \subseteq W$, we have $\dim W \geq n > m$, contradiction.

c) let $V = \mathbb{R}^2$, $W = \mathbb{R}$. then $\dim V = 2 > 1 = \dim W$.
Define $T: V \to W$ by $T(x, y) \equiv 0$.
Trivially, T is a linear transformation.
However, T is not a onto function.

7. (a) $T_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $T_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b). $T_x \circ T_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$

$T_\theta \circ T_x = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

Hence, $T_x \circ T_\theta \neq T_\theta \circ T_x$ for $\theta \in (0, \pi) \cup (\pi, 2\pi)$

(c) We have $T_x \circ T_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos 4 & -\sin 4 \\ \sin 4 & \cos 4 \end{pmatrix} = \begin{pmatrix} \cos 4 & -\sin 4 \\ -\sin 4 & -\cos 4 \end{pmatrix}$

8. $\S$2.2 #2

a) $T(e_1) = (2,3,1) = 2e_1 + 3e_2 + e_3$

$T(e_2) = (-1,4,0) = -e_1 + 4e_2 + 0e_3$

$$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$$

b) $T(e_1) = (2,1)$

$T(e_2) = (3,0) \Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

$T(e_3) = (-1,+1)$

c) $T(e_1) = 2$

$T(e_2) = 1 \Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$

$T(e_3) = -3$

9. $\S$2.2 #3: Set $v_1 = (1,2)$ and $v_2 = (2,3)$ so $\alpha = \{v_1, v_2\}$ and $w_1 = (1,1,0)$, $w_2 = (0,1,1)$, $w_3 = (2,2,3)$. so that $\gamma = \{w_1, w_2, w_3\}$.

Then

$$T(e_1) = (1,1,2) = \overbrace{-\tfrac{1}{3}w_1 + 0\,w_2 + \tfrac{2}{3}w_3}^{\text{Find by solving a linear sys.}}$$

$$T(e_2) = (-1,0,1) = -w_1 + w_2 + 0\,w_3$$

So
$$[T]_\beta^\gamma = \begin{pmatrix} 1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{pmatrix}$$

Also
$$T(v_1) = (-1, 1, 4) = -\frac{7}{3} w_1 + 2 w_2 + \frac{2}{3} w_3$$
$$T(v_2) = (-1, 2, 7) = -\frac{11}{3} w_1 + 3 w_2 + \frac{4}{3} w_3$$

So
$$[T]_\alpha^\gamma = \begin{pmatrix} -7/3 & -11/3 \\ 2 & 3 \\ 2/3 & 4/3 \end{pmatrix}$$

10. § 2.2 # 5

(a) Have $\alpha = \{E^{11}, E^{12}, E^{21}, E^{22}\}$ and
$$T(E^{ij}) = \begin{cases} E^{ij} & \text{if } i=j \\ E^{ji} & \text{if } i \neq j \end{cases} \Rightarrow [T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2E^{12}$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = E^{11} + 2E^{12}$

$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 2E^{12} + 2E^{22}$

$$\Rightarrow [T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) $T(E^{11}) = T(E^{22}) = 1$, $T(E^{12}) = T(E^{21}) = 0$

$$\Rightarrow [T]_\alpha^\gamma = (1\ 0\ 0\ 1)$$

d) $T(1) = 1 \quad T(x) = 2 \quad T(x^2) = 4$

$$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$$

e) $A = 1 \cdot E^{11} + (-2) E^{12} + 4 \cdot E^{22} \Rightarrow [A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}$

f) $[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

g) $[a]_\gamma = (a)$