1 a. Let B be a matrix in RREF that is row equivalent to A. As $\mathcal{N}(B) = \mathcal{N}(A)$, it suffices to prove the claim for B. As $m < n$, the matrix B has at least one non-pivot column, and so the associated linear system $\mathcal{LS}(B, 0)$ has at least one free variable. The sol'n to $\mathcal{LS}(B, 0)$ where all free vars have value 1 gives the needed non-zero element of $\mathcal{N}(B)$.

b. Vectors $u_1, u_2, \ldots, u_j$ in $\mathbb{R}^k$ are linearly dependent if and only if $\mathcal{N}\left(A = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_j \\ | & & | \end{pmatrix}\right)$ contains a nonzero vector. As $k < j$, this is the case by part (a).

2. a. As $S \subseteq S \cup \{v\}$, we know span(S) is contained in span$(S \cup \{v\})$. Conversely, suppose $w \in$ span$(S \cup \{v\})$, and so

$$w = a_1 u_1 + \cdots + a_n u_n + bv$$

where the u_i are in S and the a_i and b are in $\mathbb{R}$.

As $v \in \text{span}(S)$ we have $s_i \in S$ and $c_i \in \mathbb{R}$ with

$$v = c_1 s_1 + \cdots + c_m s_m$$

Thus

$$w = a_1 u_1 + \cdots + a_n u_n + b c_1 s_1 + \cdots + b c_m s_m$$

and so $w \in \text{span}(S)$. This shows $\text{span}(S \cup \{v\})$ is contained in $\text{span}(S)$, completeing the proof that $\text{span}(S) = \text{span}(S \cup \{v\})$.

b. We have $(0,1)$ and $(1,1)$ in S, and hence

$$(1,0) = (1,1) - (0,1) \text{ in } \text{span}(S).$$

By part ⓐ, have

$$\text{span}(S) = \text{span}(S \cup \{(1,0)\}) = \mathbb{R}^2$$

where the last equality is because $S \cup \{(1,0)\}$ contains both $(1,0)$ and $(0,1)$.

3. ($\Longrightarrow$) Suppose u, v are linearly dep, with $au + bv = 0$ where $a, b \in \mathbb{R}$ are not both zero. If $a \neq 0$, have $u = -\frac{b}{a} v$; if $b \neq 0$, have $v = -\frac{a}{b} u$. Either way, one of u, v is a scalar mult. of the other.

($\Longleftarrow$) If $u = av$, then $(-1) \cdot u + a \cdot v = 0$ and so u, v are linearly dependent. The case of $v = au$ is similar.

4. The statement is false as none of the vectors $\{(1, 0, 0), (0, 1, 0), (1, 1, 0)\} \subseteq \mathbb{R}^3$ are scalar mulciples of another, but they are linearly dependent with $(1, 1, 0) = (1, 0, 0) + (0, 1, 0)$

5. They are linearly independent. Suppose $a, b \in \mathbb{R}$ with

$$af + bg = 0$$

in $\mathcal{F}(\mathbb{R}, \mathbb{R})$, so in particular is the zero function. Plugging in $t = 0$ we get

$$0 = a f(0) + b g(0) = a \sin(0) + b \cos(0) = b$$

and plugging in $t = \pi/2$ we get

$$0 = a f(\pi/2) + b g(\pi/2) = a \cdot 1 + b \cdot 0 = a.$$

Thus we've shown $a = b = 0$ and so f and g are linearly independent.

6. + 7. See solutions in back of [FIS]

8. Note that W is not all of $\mathbb{R}^5$ as it does not contain $(1,2,3,4,5)$ as those coordinates sum to $15 \neq 0$. Thus $\dim W < \dim \mathbb{R}^5 = 5$, and so $\dim W \leq 4$. By trial and error, find that $\{u_5, u_6, u_7, u_8\}$ are linearly independent. Therefore, they span a subspace of W of dim 4. As $\dim W \leq 4$, this gives

$$\operatorname{span}\{u_5, u_6, u_7, u_8\} = W$$

and so $\{u_5, u_6, u_7, u_8\}$ is the basis of W we seek.

9. From [FIS] have the basis $\{E^{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$ for $M_{n\times n}(\mathbb{R})$ where

$$E^{ij} = \text{matrix which is } 0 \text{ except for the } (i,j) \text{ entry which is } 1.$$

Claim: $\beta = \{E^{ij}\}_{\substack{i=1,\dots,n\\ j=i,i+1,\dots,n}}$ is a basis for U.

Linear independence of β is immediate as it is a subset of a basis for $M_{n\times n}(\mathbb{R})$.

To see that it spans, note that if $A = (A_{ij})$ is in U, then $A_{ij} = 0$ for $j < i$ and so

$$A = \sum_{\substack{i=1,\dots,n\\ j=i,\dots,n}} A_{ij} E^{ij}$$

Thus β is a basis for U, and so

$$\dim U = \#\beta = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

10. Suppose $w_1, \ldots, w_k$ is a basis for W.

<u>Claim</u>: $\beta = \{w_1, \ldots, w_k, v\}$ is a basis for X.

If we prove the claim, we're done as then

$$\dim X = \# \beta = k+1 = \dim W + 1.$$

To see β is linearly independent, suppose

$$a_1 w_1 + \cdots + a_k w_k + bv = 0$$

for scalars a_i, b. Can't have $b \neq 0$, as then $v \in \text{span}(w_i) = W$. As $b = 0$, we get all $a_i = 0$ as the w_i's are linearly indep. So β is linearly independent.

To see β spans, note that $X = \text{span}(W \cup \{v\})$ $= \text{span}(\{w_1, \ldots, w_n\} \cup \{v\}) = \text{span}(\beta)$ as every $w \in W$ is a linear combination of the w_i's.

So β is a basis for X, completeing the problem.