## Math 416: //DRAFT ONLY// HW 11 due Wednesday, December 7, 2022.

**Important note:** This assignment is due on **Wednesday** not Friday.

**More important note:** This is the last homework assignment of the semester!

**Most important note:** There will be a combined final exam for sections B13 and C13 of Math 416, which will be held on *TBA*. Please notify me immediately if you have another exam in that timeslot.

**Webpage:** http://dunfield.info/416

**Office hours:** Here is my schedule for the rest of the semester:

**Problems:**

There is no Problem 1 on this assignment due to instructor error. *Remove this, but check numbering of solutions.*

2. Let $T$ be a *normal* operator on a finite-dimensional inner product space $V$.

   (a) Prove that $\mathcal{N}(T) = \mathcal{N}(T^*)$ and $\mathcal{R}(T) = \mathcal{R}(T^*)$.

   (b) Prove that the subspaces $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are orthogonal.

   (c) Give an example of a (non-normal) linear operator $S$ where $\mathcal{N}(S) \neq \mathcal{N}(S^*)$ and $\mathcal{R}(S) \neq \mathcal{R}(S^*)$.

   Hint: Use the following fact that you proved in HW 10: If $T$ is a linear operator on finite-dimensional inner product space $V$, then $\mathcal{R}(T^*)^\perp = \mathcal{N}(T)$ and $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$.

3. A matrix $A \in M_{n\times n}(\mathbb{R})$ is *Gramian* if there is a $B \in M_{n\times n}(\mathbb{R})$ such that $A = B^t B$. Prove that $A$ is Gramian if and only if $A$ is symmetric and all of its eigenvalues are non-negative.

   Hint: For ($\Longleftarrow$), note that $A$ is diagonalizable via an orthonormal basis $\{u_1, \ldots, u_n\}$ where $u_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$. Consider the linear operator $T$ on $\mathbb{R}^n$ where $T(u_i) = \sqrt{\lambda_i}\, u_i$. Now take $B = [T]_{\mathrm{std}}$ and check that $A = B^t B$.

4. Section 6.5 of [FIS], Problem 11.

5. Section 6.5 of [FIS], Problem 17.

6. Section 6.5 of [FIS], Problem 24.

7. Suppose $A \in M_{3\times 3}(\mathbb{R})$ is an orthogonal matrix with $\det(A) = 1$. (Recall from a prior assignment that any orthogonal matrix has determinant $\pm 1$.) In this problem, you will show $L_A$ is rotation about a line $W$ in $\mathbb{R}^3$, where $W$ passes through the origin.

   (a) First, show that any (real) eigenvalue of $A$ must be $\pm 1$.

   (b) Note that $A$ has at least one eigenvalue since its characteristic polynomial $f(t)$ has odd degree and hence at least one real root $\lambda$. In this step, you'll show that 1 is always an eigenvalue. If instead $\lambda = -1$, then $f(t) = (-1 - t)(t^2 + bt + c)$ for some $b, c \in \mathbb{R}$. Use that $\det(A) = 1$ to prove that $c < 0$ and hence by the quadratic formula that $f(t)$ splits completely over $\mathbb{R}$. Now show that the eigenvalues of $A$ are $-1$ and $1$, with algebraic multiplicies 2 and 1 respectively.

(c) Let $v_1$ be an eigenvector for $A$ with eigenvalue 1, and set $W = \text{span}(\{v_1\})$. Prove that $L_A$ preserves $W^\perp$ and acts on it by an orthogonal transformation.

(d) Use Theorem 6.23 of the text to argue that the action of $L_A$ on $W^\perp$ is by a rotation. Hint: If instead the restriction was a reflection, find a basis of $\mathbb{R}^3$ consisting of eigenvectors for $A$ which shows instead that $\det(A) = -1$.

8. Suppose $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^n$ and let $P$ be the parallelepiped spanned by them. Consider the matrix $G \in M_{n \times n}(\mathbb{R})$ where $G_{ij} = \langle v_i, v_j \rangle$. (As usual, the inner product here is just the ordinary dot product.)

(a) Show that $G$ is Gramian.

(b) Show that $\det(G) \geq 0$.

(c) Show that the unsigned volume of $P$ is $\sqrt{\det(G)}$.

In fact, $G$ is usually called the Gram matrix of these vectors.