Math 416: HW 11 due Wednesday, December 7, 2022.

Important note: This assignment is due on **Wednesday** not Friday.

More important note: This is the last homework assignment of the semester!

Most important note: There will be a combined final exam for sections B13 and C13 of Math 416, which will be held on Monday, December 12, from 8–11am in 1092 Lincoln Hall. Please notify me immediately if you have another exam in that timeslot.

Webpage: http://dunfield.info/416

Office hours: Here is my schedule for the rest of the semester:

- Monday, December 5, from 2:30–3:30pm.
- Tuesday, December 6, from 3–4pm.
- Friday, December 9, from 12–2pm.
- Sunday, December 11, from 2–4pm.

Problems:

1. Let T be a *normal* operator on a finite-dimensional inner product space V.

   (a) Prove that $\mathcal{N}(T) = \mathcal{N}(T^*)$ and $\mathcal{R}(T) = \mathcal{R}(T^*)$.

   (b) Prove that the subspaces $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are orthogonal.

   (c) Give an example of a (non-normal) linear operator S where $\mathcal{N}(S) \neq \mathcal{N}(S^*)$ and $\mathcal{R}(S) \neq \mathcal{R}(S^*)$.

   Hint: Use the following fact that you proved in HW 10: If T is a linear operator on finite-dimensional inner product space V, then $\mathcal{R}(T^*)^\perp = \mathcal{N}(T)$ and $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$.

2. A matrix $A \in M_{n\times n}(\mathbb{R})$ is *Gramian* if there is a $B \in M_{n\times n}(\mathbb{R})$ such that $A = B^t B$. Prove that A is Gramian if and only if A is symmetric and all of its eigenvalues are non-negative.

   Hint: For $(\Longleftarrow)$, note that A is diagonalizable via an orthonormal basis $\{u_1, \ldots, u_n\}$ where u_i is an eigenvector of A with eigenvalue λ_i. Consider the linear operator T on $\mathbb{R}^n$ where $T(u_i) = \sqrt{\lambda_i}\, u_i$. Now take $B = [T]_{\mathrm{std}}$ and check that $A = B^t B$.

3. Section 6.5 of [FIS], Problem 11.

4. Section 6.5 of [FIS], Problem 17.

5. Section 6.5 of [FIS], Problem 24.

6. Suppose $A \in M_{3\times 3}(\mathbb{R})$ is an orthogonal matrix with $\det(A) = 1$. (Recall from a prior assignment that any orthogonal matrix has determinant ± 1.) In this problem, you will show L_A is rotation about a line W in $\mathbb{R}^3$, where W passes through the origin.

   (a) First, show that any (real) eigenvalue of A must be ± 1.

(b) Note that A has at least one eigenvalue since its characteristic polynomial $f(t)$ has odd degree and hence at least one real root λ. In this step, you'll show that 1 is always an eigenvalue. If instead $\lambda = -1$, then $f(t) = (-1 - t)(t^2 + bt + c)$ for some $b, c \in \mathbb{R}$. Use that $\det(A) = 1$ to prove that $c < 0$ and hence by the quadratic formula that $f(t)$ splits completely over $\mathbb{R}$. Now show that the eigenvalues of A are -1 and 1, with algebraic multiplicies 2 and 1 respectively.

(c) Let v_1 be an eigenvector for A with eigenvalue 1, and set $W = \mathrm{span}(\{v_1\})$. Prove that L_A preserves $W^\perp$ and acts on it by an orthogonal transformation.

(d) Use Theorem 6.23 of the text to argue that the action of L_A on $W^\perp$ is by a rotation. Hint: If instead the restriction was a reflection, find a basis of $\mathbb{R}^3$ consisting of eigenvectors for A which shows instead that $\det(A) = -1$.

7. Suppose $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^n$ and let P be the parallelepiped spanned by them. Consider the matrix $G \in M_{n\times n}(\mathbb{R})$ where $G_{ij} = \langle v_i, v_j \rangle$. (As usual, the inner product here is just the ordinary dot product.)

   (a) Show that G is Gramian.

   (b) Show that $\det(G) \geq 0$.

   (c) Show that the unsigned volume of P is $\sqrt{\det(G)}$.

   In fact, G is usually called the Gram matrix of these vectors.