
Webpage: http://dunfield.info/416

Office hours: Wednesday 2:30–3:30pm and Thursday 3:00–4:00pm; other times possible by appointment. My office is 378 Altgeld.

Problems:

1. Let \( \mathbb{C} \) denote the field of complex numbers, as discussed in detail in Appendix D of [FIS]. As with any field, we can consider vector spaces, linear transformations, and matrices over \( \mathbb{C} \) rather than over our usual field \( \mathbb{R} \).
   
   (a) The complex numbers \( \mathbb{C} \) can be viewed as a vector space over either \( \mathbb{C} \) or \( \mathbb{R} \) with the usual scalar multiplication. Prove that \( \mathbb{C} \) has dimension 1 as a vector space over \( \mathbb{C} \) but has dimension 2 as a vector space over \( \mathbb{R} \). In each case, give an explicit basis.
   
   (b) Since \( \mathbb{R} \) is a subset of \( \mathbb{C} \), if \( V \) is a vector space over \( \mathbb{C} \) then it is also a vector space over \( \mathbb{R} \): just use the same scalar multiplication but restricted to scalars in \( \mathbb{R} \). If \( V \) has dimension \( n \) as a vector space over \( \mathbb{C} \), prove that it has dimension \( 2n \) as a vector space over \( \mathbb{R} \). Hint: Use Theorem 2.19 from [FIS] to reduce to the case where \( V \) is just \( \mathbb{C}^n \).
   
   (c) Diagonalize the following matrices over \( \mathbb{C} \) by giving a \( Q \in M_{2\times2}(\mathbb{C}) \) so that \( Q^{-1}AQ \) is diagonal.

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}
\]

2. Section 5.1 of [FIS], Problem 1.

3. Section 5.2 of [FIS], Problem 1.

4. Section 5.2 of [FIS], Problem 2, parts (e) and (g).

5. Section 5.2 of [FIS], Problem 3, parts (a) and (d).

6. Prove that similar matrices have the same characteristic polynomial.

7. Section 5.2 of [FIS], Problem 7.

8. If \( A \) is a square matrix prove that \( A \) and \( A^t \) have the same eigenvalues. Do they have the same eigenvectors? Either prove they do, or give a counterexample.

9. Suppose that \( A \) in \( M_{n\times n}(\mathbb{R}) \) has two distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \), and that \( \dim(E_{\lambda_1}) = n - 1 \). Prove that \( A \) is diagonalizable.

10. Section 5.3 of [FIS], Problem 6. 5th edition changed wording and time period from months to weeks. Copy one version here.