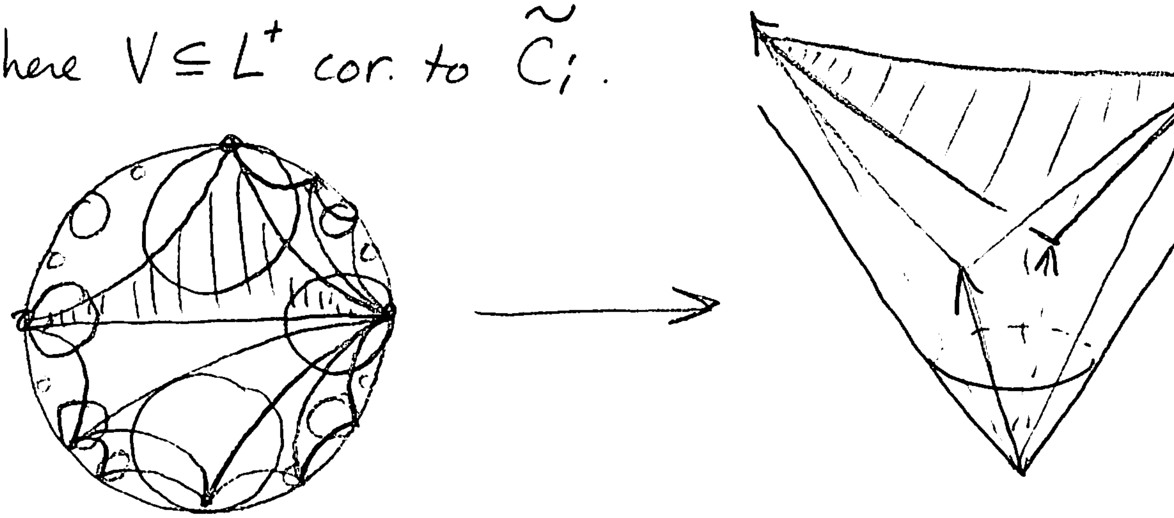
# Lecture 9: Finding the canonical triangulation



Suppose $S$ is a hyp surface with any geodesic ideal tri. $\mathcal{T}$ and fixed cusp nbhds $C_i$.

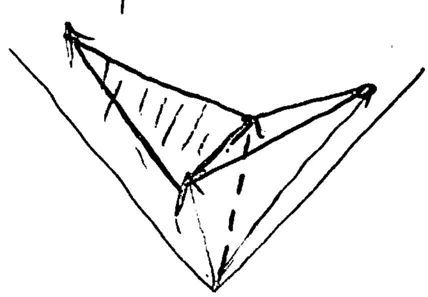
Have a $\pi_1 S$ equiv map $\widetilde{\mathcal{T}} \longrightarrow \mathbb{R}^{2,1}$ sending each ideal tri to a linear one w/ verts in $V$, where $V \subseteq L^+$ cor. to $\widetilde{C}_i$.



($\star$)

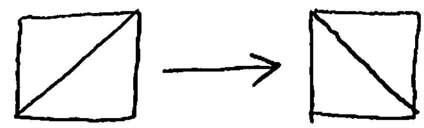
Gives a "bent plane" $X$ inside $L^+$ that projects out to the orig. geod ideal tri.

The orig. tri. is canonical $\Longleftrightarrow$ $X = \partial P$, $P =$ convex hull of $V$. $\Longleftrightarrow$ $X$ bounds a convex region inside $L^+$ $\Longleftrightarrow$ each pair of tri meeting along an edge "fold up".



not down as in ($\star$)

Purely local and only need check for one edge of $X$ in each $\pi_1 S$-orbit, i.e. once for each edge of $\mathcal{J}$ downstairs. [Aside about Penner coor, horoball decorations... ]

If pair folds down, do a move  along this edge. to create $\mathcal{J}_1$.

Get a seq $\mathcal{J}_1, \mathcal{J}_2, \ldots$ with cor $X_1, X_2, \ldots$ inside $L^+$. that "move down."


Prop: This must terminate.

An edge class is a segment joining two elts in $V$, together with its $\pi_1 S$ orbit.

Claim: Only finitely many edge classes below the orig. $X$.

Only finitely many vertex classes, so suffices to bound the number of edge classes ending at a fixed $v_0 \in V$.

Consider $f : X \to \mathbb{R}$ by $f(x) = \langle x, x \rangle$ which is cont. and $\pi_1 S$ equiv. It is bounded since

$\pi_1 S \backslash X =$ compact. of $S$ adding one pt per cusp

Say $f(X) \subseteq [-r^2, 0]$


<u>Lemma:</u> $u, v \in L^+$. Then

a) $\langle u-v, u-v \rangle = -2\langle u, v \rangle \geq 0$

b) $\min \langle x, x \rangle$ for $x = (1-t)u + tv$ $t \in [0,1]$ is at the midpt $\langle \frac{u+v}{2}, \frac{u+v}{2} \rangle = \frac{\langle u, v \rangle}{2}$

c) The dist between the horocircles $H_u$ and $H_v$ is $\log\left(\frac{\langle u, v \rangle}{-2}\right)$.

<u>Pf</u>: a) is because $u - v$ is space-like.

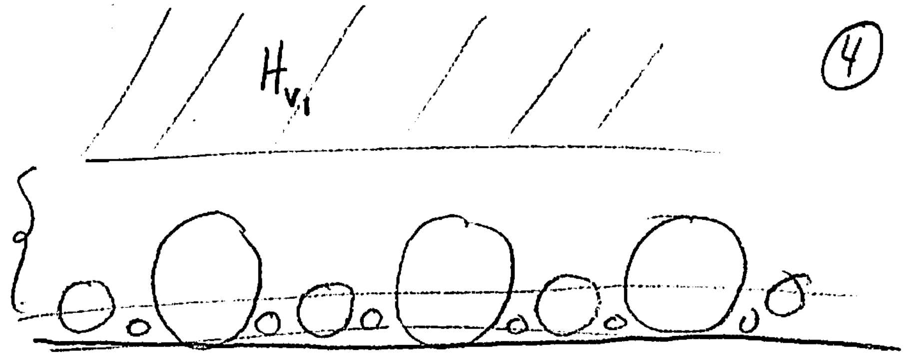
b) is since $\langle x, x \rangle = 2(1-t)t\langle u, v \rangle$ and $\langle u, v \rangle \leq 0$.

c) Transform to reduce to $u = (t, 0, t)$ and $v = (-t, 0, t)$ and see $\operatorname{dist}(H_u, H_v) = 2\log t = \log t^2 = \log\left(\frac{\langle u, v \rangle}{-2}\right)$. ▨

<u>Pf of claim</u>: An edge $e$ from $v_0$ to $v_1$ lying below $X$ must sat $\langle x, x \rangle \geq -r^2$ along the seg $(v_0, v_1)$

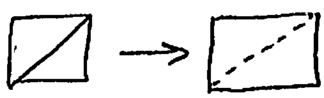
$\Rightarrow \frac{\langle v_0, v_1 \rangle}{2} \geq -r^2 \Rightarrow \operatorname{dist}(H_{v_0}, H_{v_1}) \leq \log r^2$.

Now there are only finitely many orbits of such horoballs $H_{v_1}$:




and hence finitely many such edge classes.

This proves the claim and hence the prop. ∎

<u>Remarks</u>: 1) To get final cellulation have to erase any edges  where there is 0 fold.

2) For $n=2$, the above proves the lemma about finiteness of faces of the canonical cellulation that I skipped last time.

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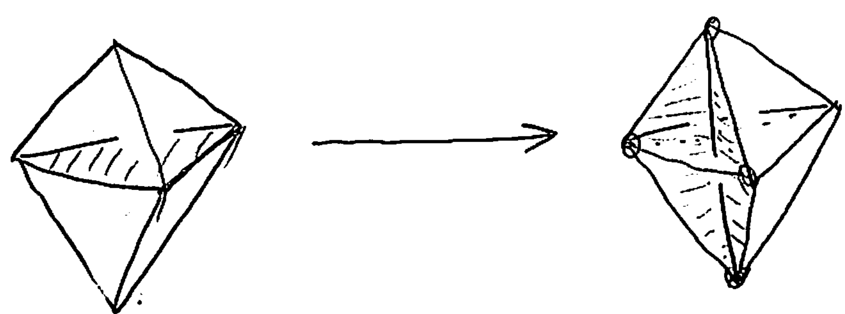
Case of 3-mflds is very similar, with flip replaced by $2 \to 3$ and $3 \to 2$ moves.



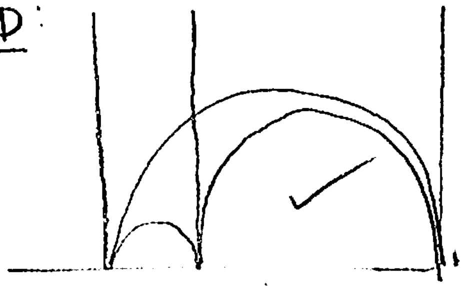
Still a local test on faces and valence 3 edges, depending on whether things are

Concave/convex in $\mathbb{R}^{3,1}$ inside the light cone.

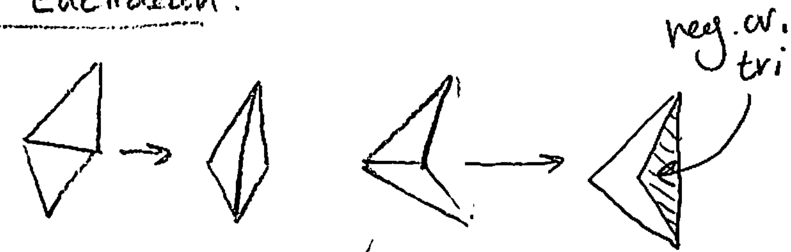
For details, see Week's paper.

New issue: creation of neg. orient tets.

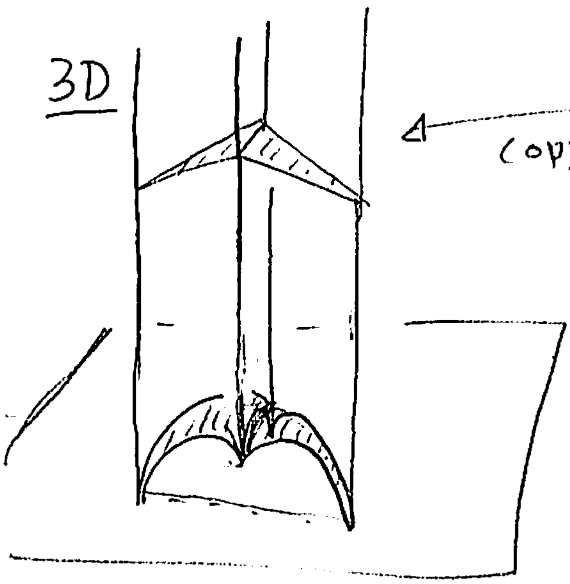
2D:



2D Euclidean:




3D



copy into horosphere.

Now the "move down" algorithm can get stuck.