Lecture 8: More on canonical cellulations

Last time: $\mathbb{H}^n \subseteq \mathbb{R}^{n,1}$ $\quad L = \{\langle x,x\rangle = 0\}$ light cone

For $v \in L$ with $v_{n+1} > 0$, the set
$$H_v = \{x \in \mathbb{H}^n \mid -1 \leq \langle x, v\rangle\} \text{ is a horoball.}$$

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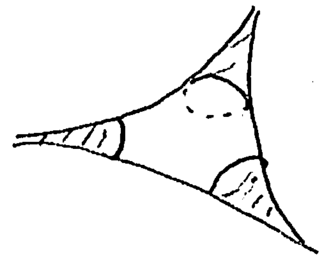
M^n hyp w/ finite-vol and cusps.



Pick cusp nbhds of equal vol

Get $\pi_1 M$-equiv. horoballs in $\mathbb{H}^n$

V = cor. vectors in L; discrete and $\pi_1 M$ equiv.

P = convex hull of V, ∂P has a cell str

D = project ∂P out to $\mathbb{H}^n$ through 0 to a decomposition of $\mathbb{H}^n$ into ideal polyhedra.

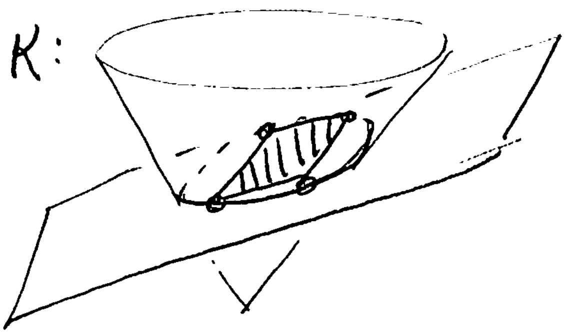
Lemma: Each cell of D has finitely many vertices and there are only finitely $\pi_1 M$ orbits of cells.

Pf: See [Epstein-Penner 1988].

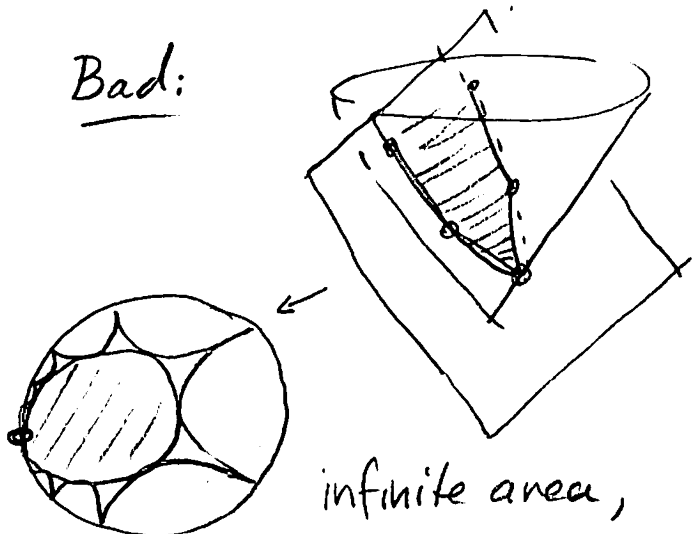
It is here that $\text{vol}(M) < \infty$ is used.

Some worries for $n=2$. Suppose E is the plane containing the face:

OK:



Bad:

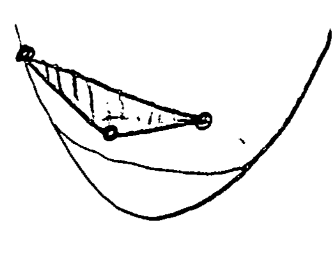


infinite area, but stab could be para fixing p.

<u>Lemma</u>: The action of $\pi_1 M$ on the cells of D is free.

<u>Pf</u>. Suppose a cell C with verts $v_1, \ldots, v_k$ is pres. by $\gamma \in \pi_1 M$. Then $b = \frac{1}{k}(v_1 + \cdots + v_k)$ and $\bar{b} = \dfrac{b}{\sqrt{-\langle b, b\rangle}}$ are pres by γ. Since $\pi_1 M$ acts freely, $\gamma = id$.

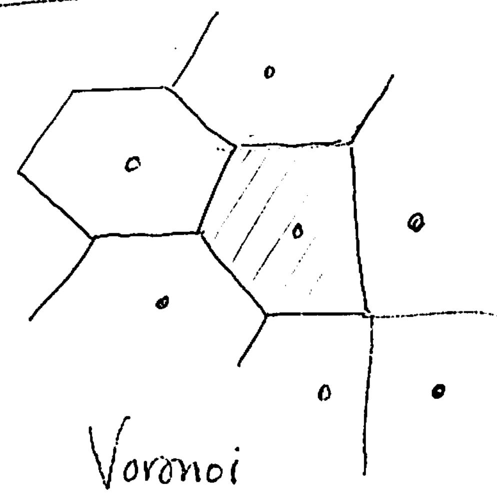
<u>Note</u>: Gives <u>barycentric coordinates</u> to hyperbolic poly (ideal or otherwise).

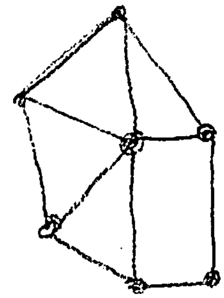


[Remark on sphereical / hyp. triangles.]

The induced cell str $\bar{D}$ on M is called the canonical cellulation. [Does not dep. on init choice of cusps.]

Dual viewpt: Ford domain.

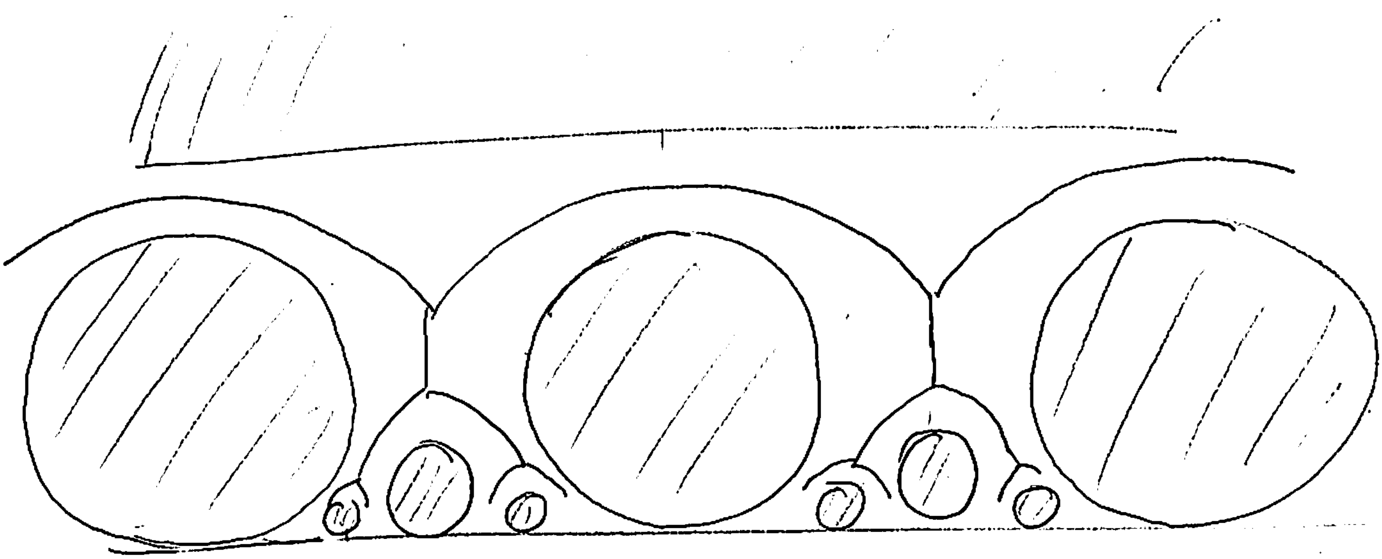


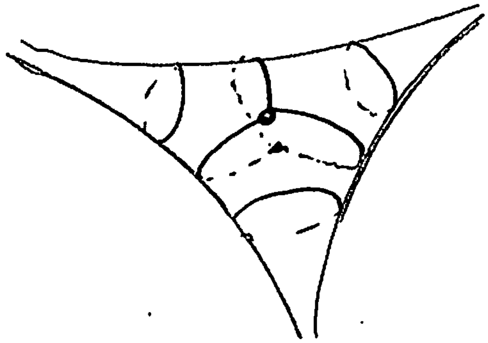


"Typically" a triang.

Works for any geom but not very canonical.

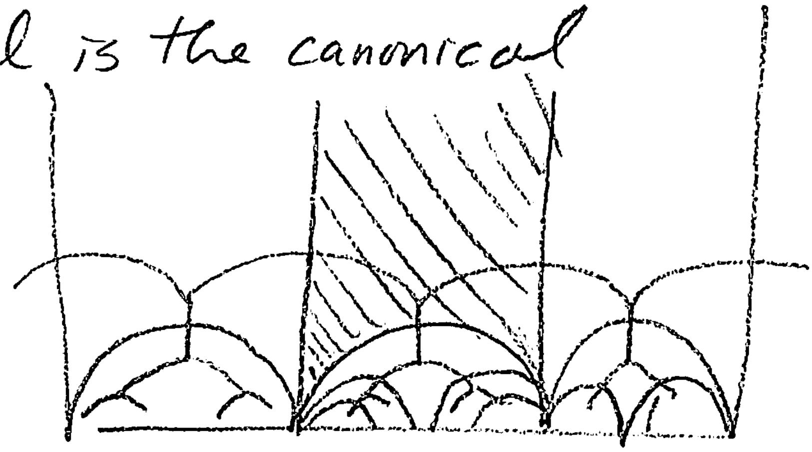
For hyp M with fixed cusp nbhds, take cells consist of pts closest to each nbhd.



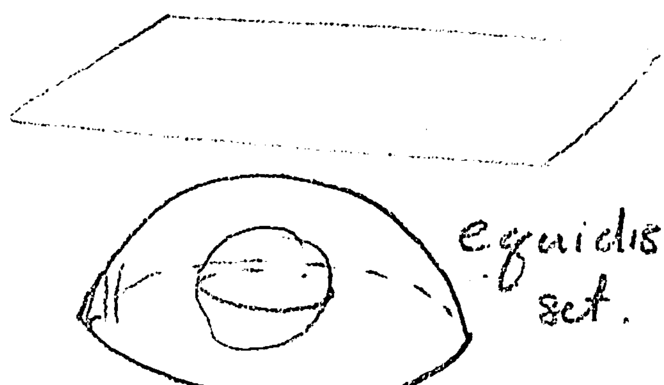


This is the Ford decomp.
Note cells upstairs have parabolic stab.

The Delaunay dual is the canonical cell decomposition.

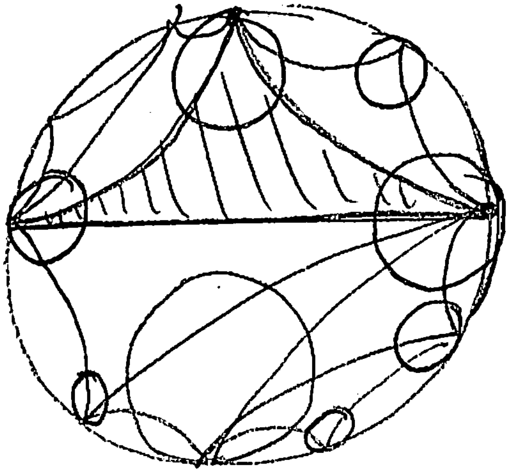


[SnapPy pictures]

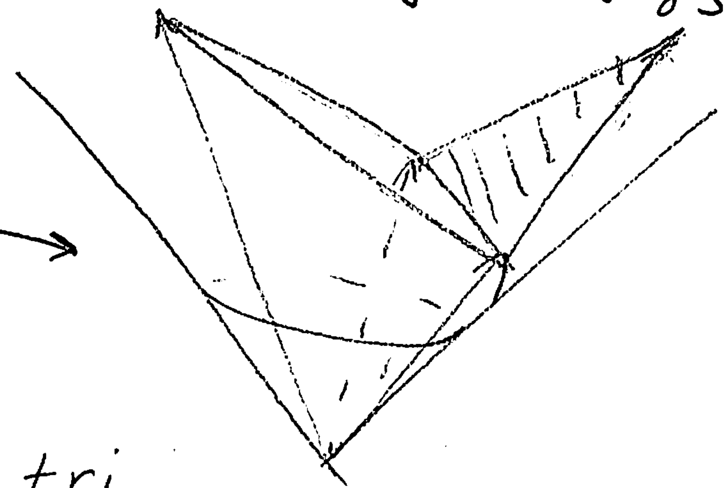


equidist set.

How to find the canonical cell decomp? Suppose our surface S has some init ideal tri $\mathcal{T}$ and cusp nbhds.
Have a $\pi_1 S$ equiv map $\tilde{\mathcal{T}} \longrightarrow \mathbb{R}^{2,1}$ sending each ideal tri to the linear one with verts in V.



horoballs $\leftrightarrow$ conj of π_1 (component of ∂S)

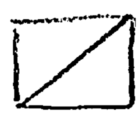
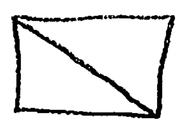


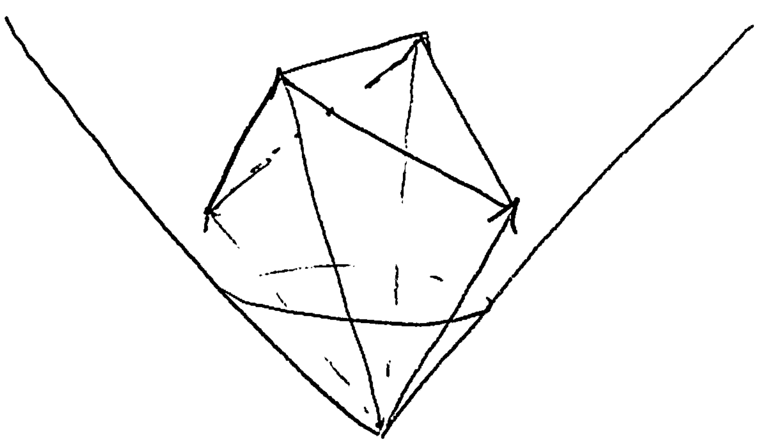
This is the canonical tri

when image is ∂P where $P =$ Convex hull of V.

$\Longleftrightarrow$ image bounds a convex region

$\Longleftrightarrow$ each pair of tri meeting along an edge "fold up"

If a pair fold down, do a  $\longrightarrow$  move.



Purely local check, leads to algorithm...