Lecture 7: Canonical cell decompositions.

Previously... $\mathbb{R}^{n,1} = \mathbb{R}^{n+1}$ with $\langle x,y\rangle = x^t J y$ for $J = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$.

$\mathbb{H}^n = \{ x \in \mathbb{R}^{n,1} \mid \langle x,x\rangle = -1 \text{ and } x_{n+1} > 0 \}$

with R-metric $= \langle\, ,\rangle|_{T_p\mathbb{H}^n}$. Then $\mathrm{Isom}^+(\mathbb{H}^n)$ is

$SO_0(n,1) = \{ A \in SL_n\mathbb{R} \mid A^t J A = J \text{ and } A \text{ pres } \mathbb{H}^n \}$

Geodesics are (plane through 0) $\cap\, \mathbb{H}^n$.

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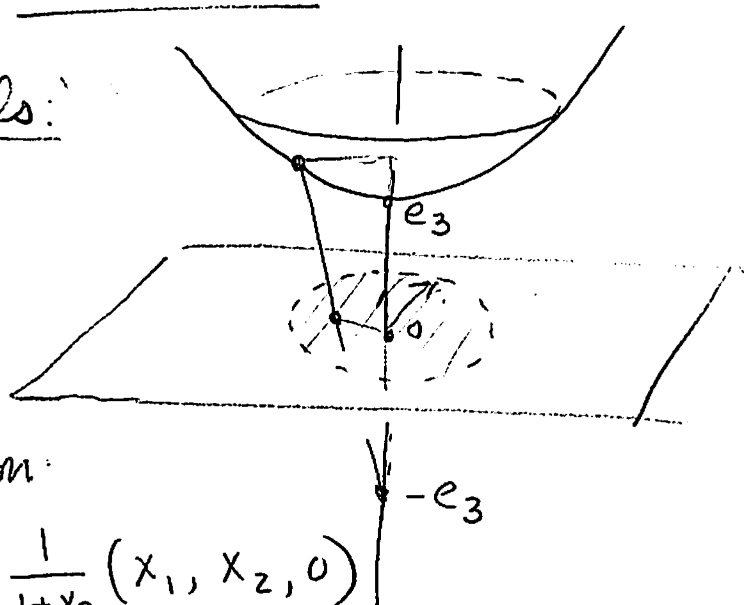


Connections to other models:

Poincaré disk:

$D = \{ (x_1, x_2, 0) \mid x_1^2 + x_2^2 < 1 \}$

Use Sterographic projection:

$\mathbb{H}^2 \to D \quad (x_1, x_2, x_3) \longmapsto \frac{1}{1+x_3}(x_1, x_2, 0)$

Can check this is an isom.

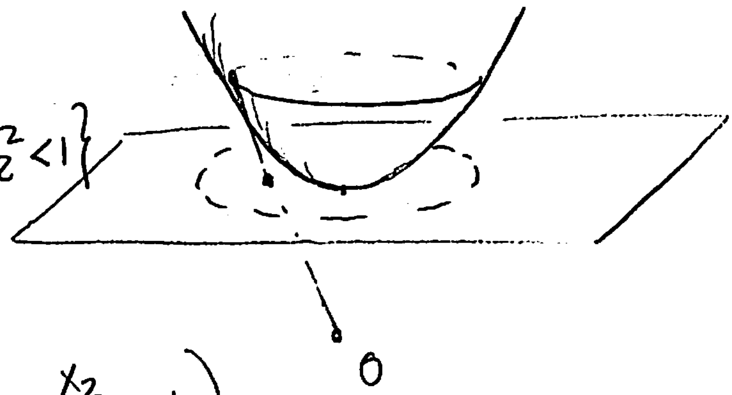
Note: $\mathbb{H}^n$ has const curve as it is isotropic. The constant is -1 as:

$$\mathrm{len}\left(\begin{smallmatrix}\text{circle of}\\ \text{rad } r\end{smallmatrix}\right) = 2\pi \sinh r = 2\pi\left(r + \frac{1}{6} r^3 + O(r^5)\right)$$

$$= 2\pi\left(r - \frac{K}{6} r^3 + O(r^4)\right)$$

Klein model:

$K = \{(x_1, x_2, 1) \mid x_1^2 + x_2^2 < 1\}$



Stero. proj from 0

$(x_1, x_2, x_3) \to \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right).$

Geodesics are Euclidean straight lines, isometries are projective transformations.

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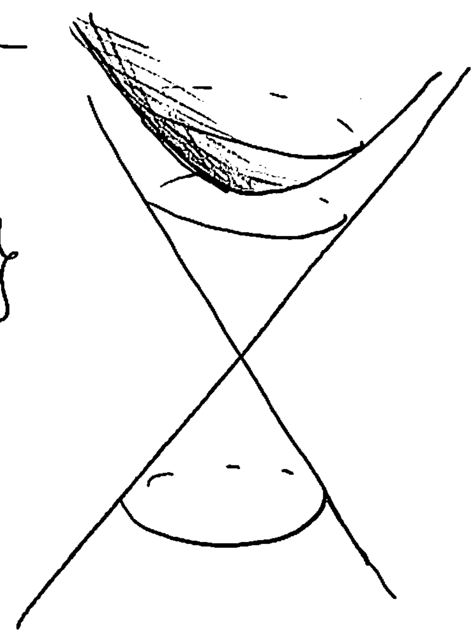
Sphere at infinity:



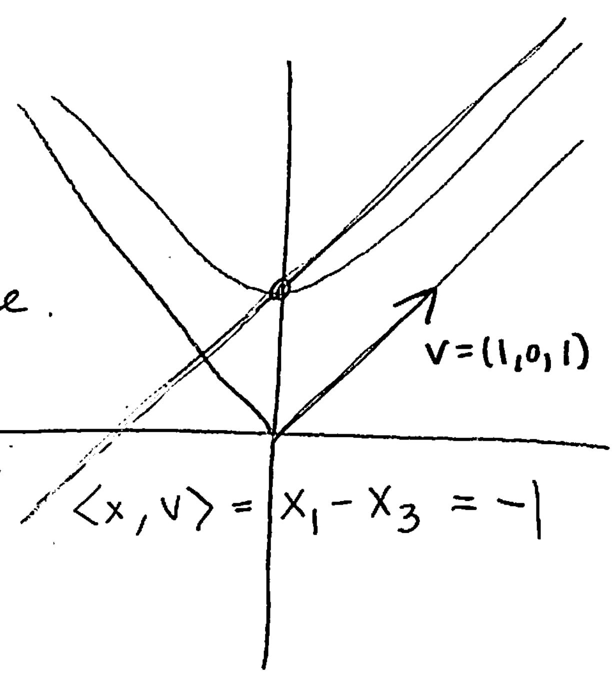
L = light cone $= \{x \in \mathbb{R}^{n,1} \mid \langle x, x\rangle = 0\}$

$S^{n-1}_\infty = \{x \in L \mid x_{n+1} > 0\} / \mathbb{R}_{>0}.$

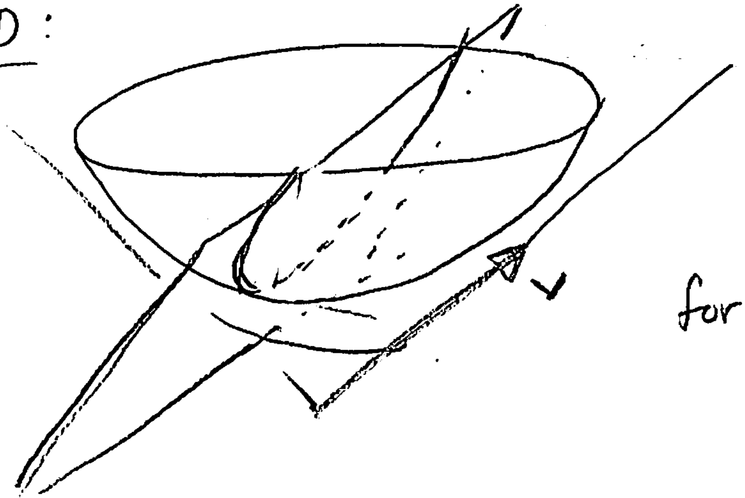
Horoball: $v \in L$ with $x_{n+1} > 0$.

Set $H_v = \{x \in \mathbb{H}^n \mid -1 \leq \langle v, x\rangle\}$

and $S_v = \{x \in \mathbb{H}^n \mid \underbrace{-1 = \langle v, x\rangle}_{\text{eqn of a plane}}\}$.



In 3D:



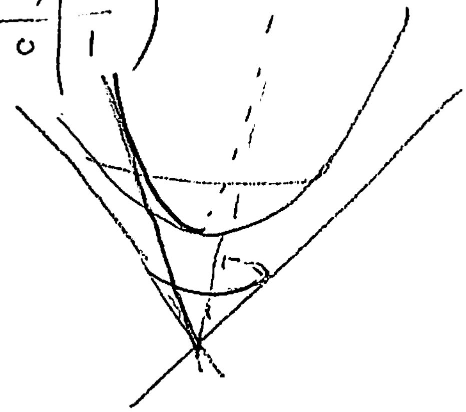
Curve is a parabola:

$(t^2, \sqrt{2}\cdot t, t^2+1)$

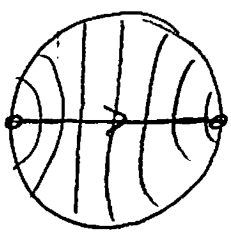
for $v=(1,0,1)$

3 Kinds of Isom (for any $\mathbb{H}^n$):

elliptic: fix a pt in $\mathbb{H}^n$, e.g. $\left(\begin{array}{cc|c} \multicolumn{2}{c|}{O(2)} & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline 0 & 0 & 1 \end{array}\right)$



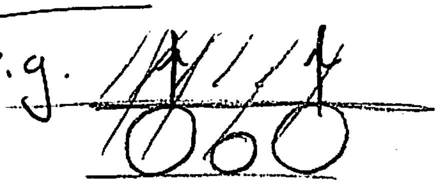
hyperbolic: no fixed pt in $\mathbb{H}^n$, two on S^{n-1}_∞; e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}$



Have an axis, etc.

parabolic no fixed pt in $\mathbb{H}^n$, one on S^{n-1}_∞,

e.g.



$z \mapsto z+t$

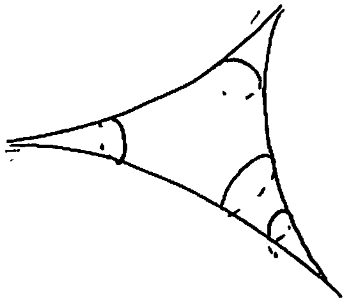
A horosphere is the orbit of a pt under the parab. that fix a pt on S^{n-1}_∞.

e.g. $A_t = \begin{pmatrix} 1-t^2 & \sqrt{2}t & t^2 \\ -\sqrt{2}t & 1 & \sqrt{2}t \\ -t^2 & \sqrt{2}t & 1+t^2 \end{pmatrix}$ in $SO_0(2,1)$.

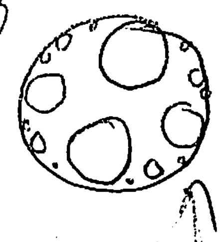
are the parabolics fixing $v=(1,0,1)$

and S_v is the orbit of e_3 under A_t.

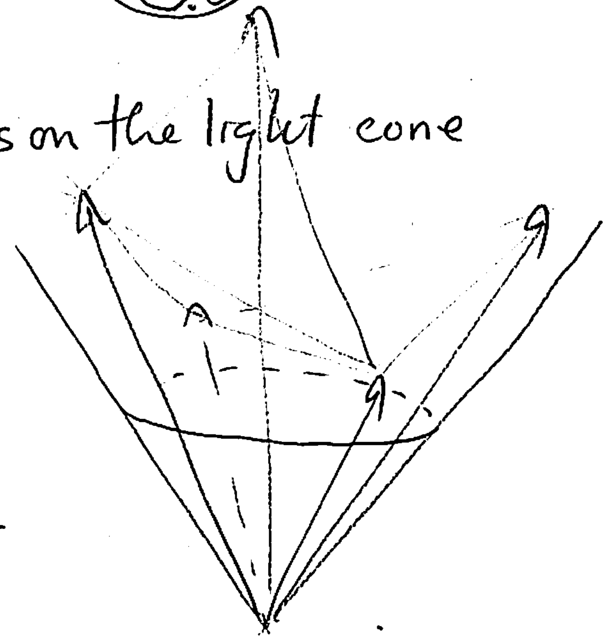
Suppose S is a hyp. surface of finite area w/ cusps. Choose disjoint cusp nbhds with equal area.



Gives a $\pi_1 S$-equivariant collection of horoballs.

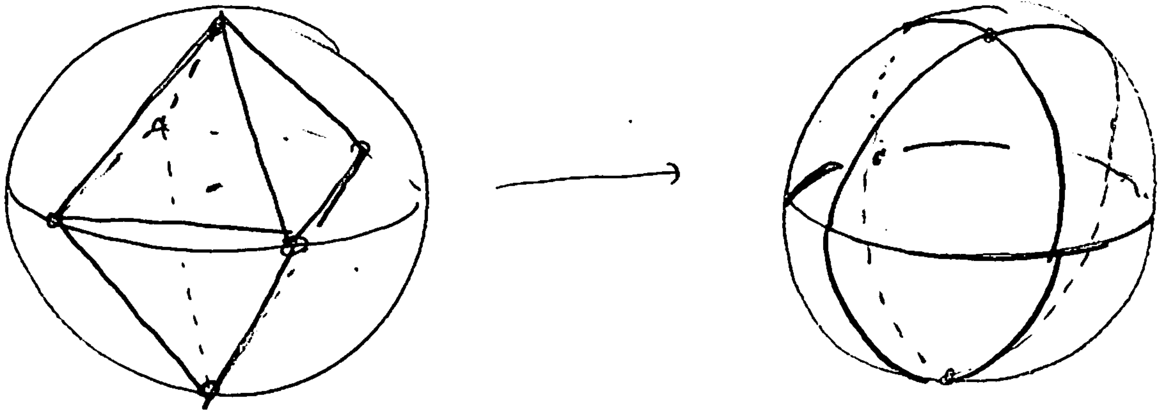


Let V be collection of vectors on the light cone cor. to these. Note that $\forall C$, $\{v \in V \mid v_3 \leq C\}$ is finite: if V accumulates somewhere, can assume its at $(1,0,1)$ and then all $w, w' \in L$ near V have $H_w \cap H_{w'} \neq \emptyset$ $\rightsquigarrow$ violates that cusp nbhds are disjoint. (Note: small $v \in L \leftrightarrow$ large horoball).



Let P be the (ordinary) convex hull of V in $\mathbb{R}^3$, and now project ∂P onto $\mathbb{H}^2$ via lines through 0. The result is a decomp D of $\mathbb{H}^2$ into ideal polyhedra, since the edges in the cellulation of ∂P turn into geodesics.

Compare with S^2:



Note that our $\pi_1 S$ action pres V and hence D. Moreover, no elt $\gamma \in \pi_1 S$ sends a cell C of dim > 0 of D to itself: If C comes from $v_1, v_2, \ldots, v_K$ in V, then $v_b = \frac{1}{K}(v_1 + \cdots + v_K)$ is fixed by γ. and hence $\dfrac{v_b}{\sqrt{-\langle v_b, v_b \rangle}}$ in $\mathbb{H}^2$ is fixed by γ, contradicting that $\pi_1 S$ acts freely.

Thus D decends to a cellutation $\overline{D}$ of S, the canonical Epstein-Penner decomposition. $\blacksquare$