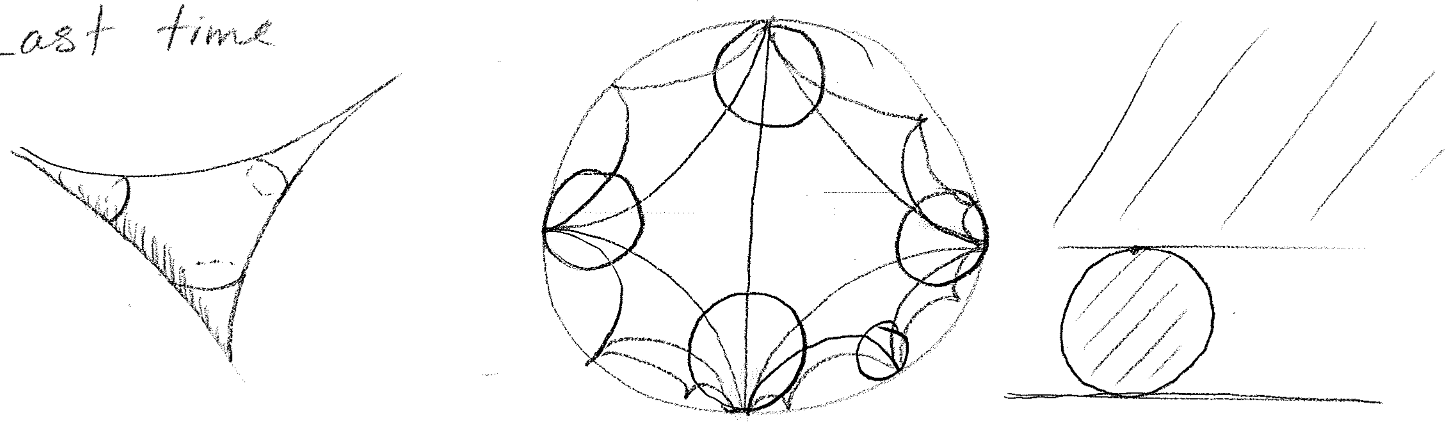
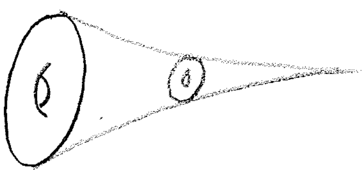
# Lecture 3:

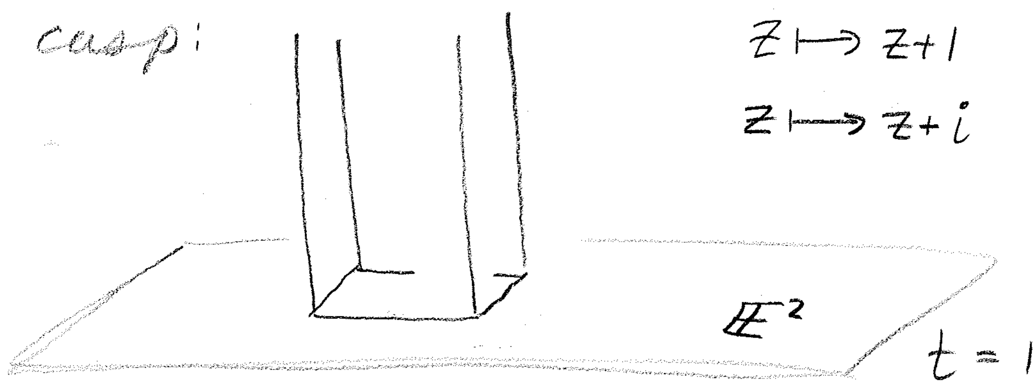
Last time



If $M^3$ is a complete hyp. mfld of finite volume, each end is a cusp:



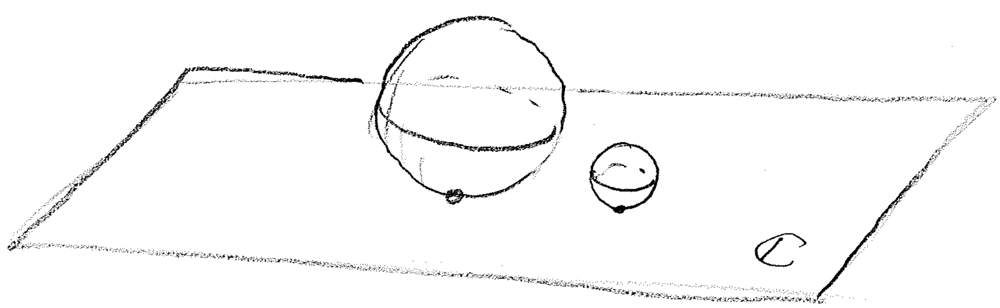
$(x, y, t)$




$z \mapsto z+1$

$z \mapsto z+i$

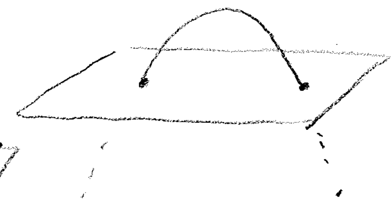
$\frac{1}{t^2} g_{\mathbb{E}^3}$




Horoball: region $t \geq t_0$ (or image under isom)

Horosphere: $\{t = t_0\}$ has a Euclidean metric.

Notes: 1) Horoballs are convex



2) Cusp torus has a shape 

Moreover, there are finitely many cusps. Hence $M$ is the interior of a compact 3-mfld with $\partial M$ a union of tori.

Ex: $K$ a knot ($S^1 \hookrightarrow S^3$). Knot complement $\overbrace{C}^{\text{noncpt}} = S^3 \setminus K$

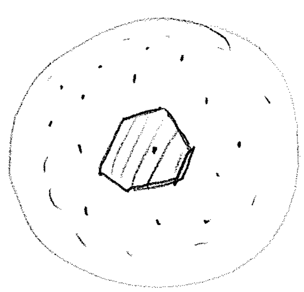
contains the knot exterior $E = S^3 \setminus \mathring{N}(K)$,

and $C \cong E \setminus \partial E$.

SnapPy interlude: Borr. Rings: Horoball diag.
Dirichlet domains

Dirichlet Domain: $\Gamma = \pi_1 M \subseteq \mathrm{Isom}^+(\mathbb{H}^3)$ Pick $x_0 \in \mathbb{H}^3$. Set $D_{x_0} = \left\{ x \in \mathbb{H}^3 \mid d(x, x_0) \leq d(\gamma \cdot x, x_0) \text{ for all } \gamma \text{ in } \Gamma \right\}$

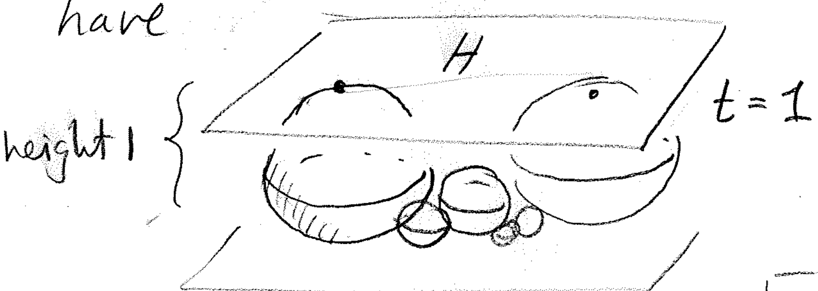
a Vornoi cell for $\Gamma \cdot x_0$.



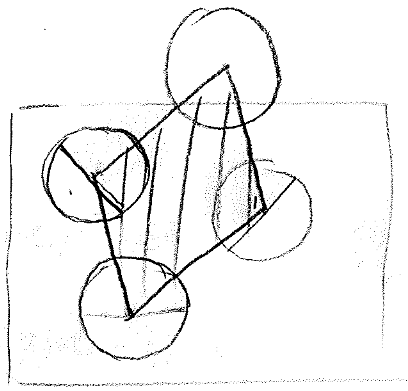
Collectively, these tile $\mathbb{H}^3$, invariant under $\Gamma$.

Thm: Suppose $M^3$ is a finite-volume hyp. with cusps. Then $vol(M) \geq \sqrt{3}/4$.

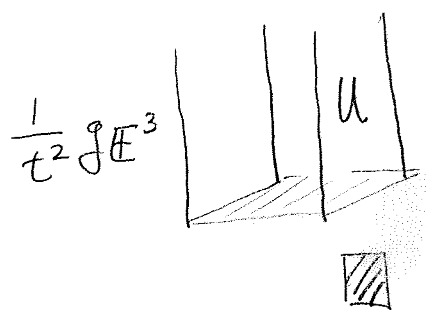
Pf: Expand out a collection of disjoint cusps untill they bump into themselves or each other. In $\mathbb{H}^3$ have




Viewed from above see disjoint unit diam circles.



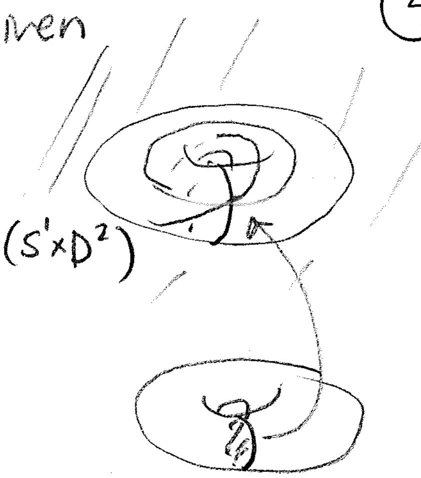
Let $\Lambda = \{\gamma \in \pi_1 M \mid \gamma$ pres this horosphere$\}$ so $T = \Lambda \backslash H =$  Note every elt of $\gamma$ trans. a dist of $\geq 1$ in the Euclid metric on $H \Rightarrow$ Area$(T)$ is $\geq \sqrt{3}/2 \Rightarrow$ Vol(cusp) =




$\iiint_U t^{-3}\,dx\,dy\,dt = \frac{\text{Area}(T)}{2}$.

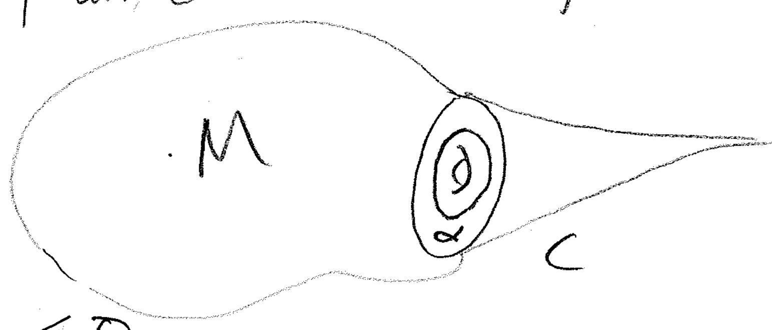
Can do better using a holoball packing argument.

Dehn filling: $N^3$ with $\partial N =$ (torus). Given an ess. simple closed curve $\alpha$ on $\partial N$ let $N(\alpha) = N \cup_f S^1 \times D^2$ where $f : \partial N \to \partial(S^1 \times D^2)$ takes $\alpha$ to $pt \times \partial D$



Thurston's Hyp Dehn Filling Thm: $M^3$ comp finite-vol hyp with one cusp. Then all but finitely many Dehn fillings on $M$ (i.e. on $N$ with $\mathrm{int}(N) = M$) are hyp. [Now at most 10 exceptions]

Gromov-Thurston $2\pi$-Thm: Suppose $\alpha$ has len $> 2\pi$ on the boundary of an embedded cusp nbhd $C$. Then $M(\alpha)$ has a Riemannian metric of sectional curve $< 0$.




Note: Combined with previous thm, this gives a weak version of HDFT with an explicit upper bound.

<u>$2\pi$-idea:</u> Metric on $(M \setminus \operatorname{int}(C)) \subseteq M(\alpha)$ is the original one. Extend explicitly to the added solid torus $S$ using cylindrical coor, calc that curv $< 0$.

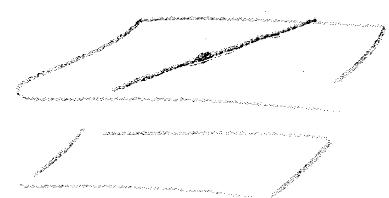
Why $2\pi$? Let $D$ be a least area disc in $S$ with $\partial D$ a geod rep $\alpha$ in $\partial S$ (a flat torus).

Now Gauss-Bonnet says

$$2\pi \underbrace{\chi(D)}_{1} = \int_D K \, dA + \int_{\partial D} k_g \, ds$$

$\int_D K \, dA$: neg. if $M(\alpha)$ has a metric of neg. curv as $K_p(D) \leq K_p(M(\alpha))$ as $D$ is minimal.

$\int_{\partial D} k_g \, ds$ $= \operatorname{len}(\partial D)$ by calc. the geod. curv of a line in a horosphere



$$\Longrightarrow 2\pi \geq \operatorname{len}(\partial D).$$