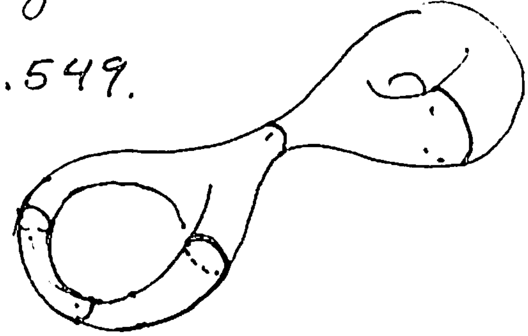
Lecture 18: Proof by parameter space.

[Gabai-Trnkova 2013] Except for Vol3, every closed hyp. 3-mfld has a simple geodesic with tube radius $\geq \log(3)/2 \geq 0.549$.



[Gabai-Meyerhoff-N. Thurston 2001]
There are exactly 7 clsd hyp 3-mflds where the shorest geod has tube rad $< \frac{\log 3}{2}$.

Reminder: · Used to show $\mathrm{Diff}(M) \simeq_{h.e.} \mathrm{Isom}(M)$.
· Show the Weeks mfld is that of smallest vol.

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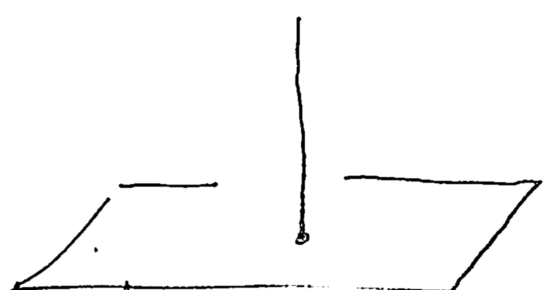
Marked groups: $\{G, f, w\}$ with $G \leq \mathrm{Isom}^+ \mathbb{H}^3 = PSL_2\mathbb{C}$

$G = \langle f, w \rangle$, f hyperbolic with axis A_f, $wA_f \cap A_f = \emptyset$.

Claim:
$$\dim_{\mathbb{C}} \left(\text{Such } \{G, f, w\} \big/ \text{conj by } PSL_2\mathbb{C} \right) = 3.$$

Idea: Normalize so $A_f = (0, \infty)$,

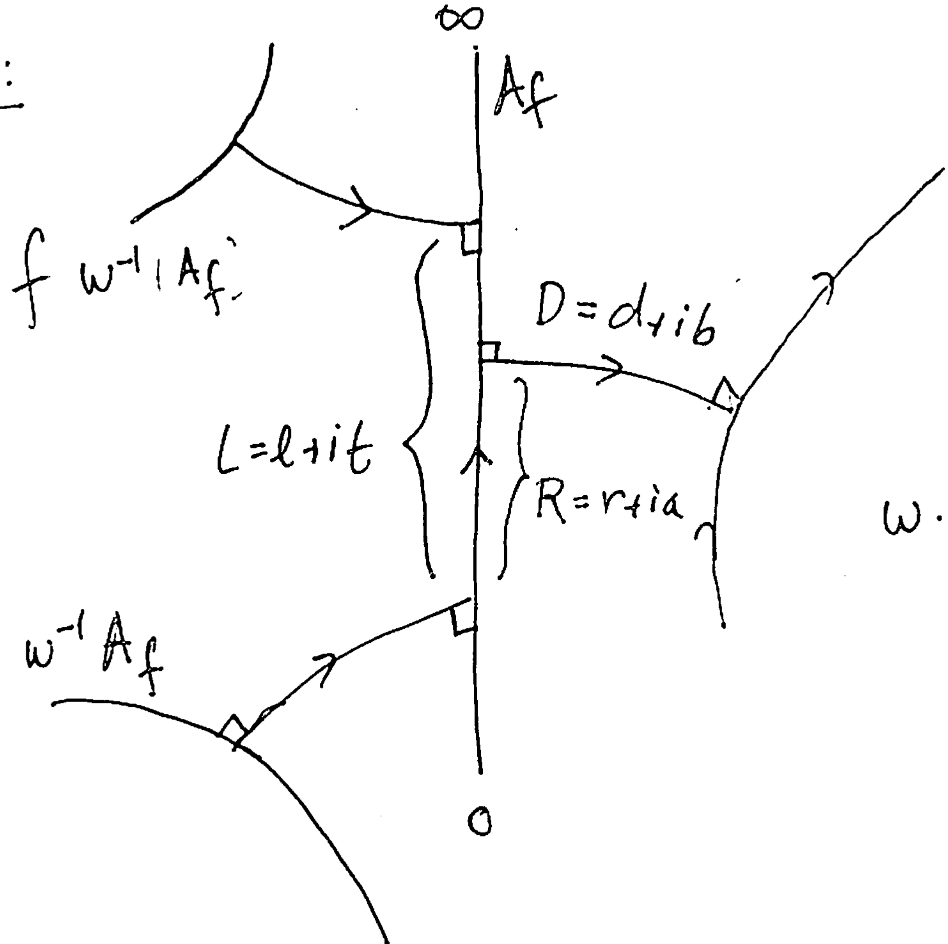


$f = \begin{pmatrix} a e^{i\theta} & 0 \\ 0 & \frac{1}{a} e^{-i\theta} \end{pmatrix}$ $\quad z \longmapsto a^2 e^{2i\theta} z$

gives 1 param. The conj stab of of f in $PSL_2\mathbb{C}$ is $\cong \mathbb{C}^\times$ so there are

$\dim_{\mathbb{C}} PSL_2\mathbb{C} - 1 = 2$ param for w.

Concretely:



$$w \cdot A_f = A_{wfw^{-1}}$$

Here $l + it$ means hyp len l and twist by t radians.

$$\mathcal{P}' = \{(L, D, R) \in \mathbb{C}^3 \mid \mathrm{Re}(L) > 0,\ \mathrm{Re}(D) > 0\}$$

$\mathcal{T}' \subseteq \mathcal{P}'$ the subset where

1) Relength(f) is minimal among all non-triv. elts of G.

$\quad$ ↖ $\inf\{d(x, f \cdot x) \mid x \in \mathbb{H}^3\}$

2) $w \cdot A_f$ is a <u>closest</u> trans. of A_f under G.

3) Ortholines as shown (point: replacing w with $w' = w f^n$ doesn't change $w A_f$ but does change $w^{-1} A_f$)

Prop: Any $\{G, f, w\} \in \mathcal{J}'$ is discrete, torsion-free, and parabolic free.

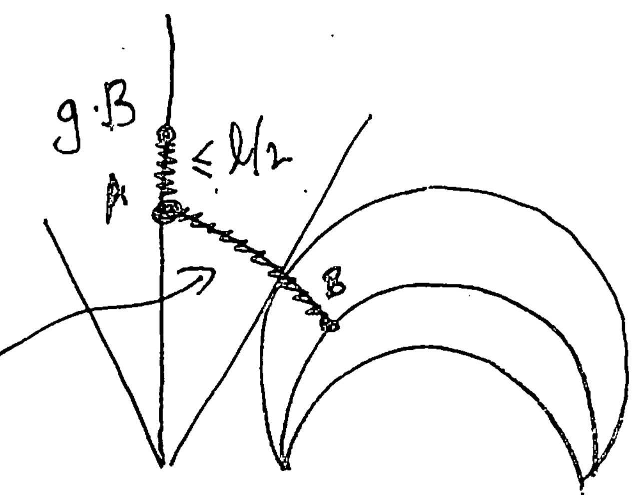
Pf: G has no elliptic or parabolic elts as these have $\mathrm{Relength}(g) = 0$. If G were indiscrete, $\exists\, g_i \neq 1$ in G with $g_i \to 1 \Rightarrow \mathrm{Relength}(g_i) \to 0$. ▨

Prop: All clsd geod of length < 0.09 in any hyp M^3 have tubes with rad $> \ln(3)/2$. Same when len > 1.3.

$\mathcal{P}$ = subset of $\mathcal{P}'$ where
$$\left.\begin{array}{l} 0.09 \le l \le 1.3 \\ l/2 \le d \le \ln(3) \\ 0 \le r \le l/2 \\ -\pi \le a, b, t \le \pi \end{array}\right\} \text{compact!}$$

Study: $\mathcal{J} = \mathcal{P} \cap \mathcal{J}'$

Idea:



$< l/2$

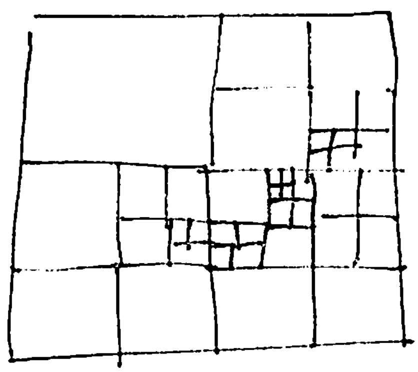
$d(g\cdot B, B) < l$ volating (1).

Suppose $\{G, f, w\} \in \mathcal{P}$. A *killerword* W in FreeGroup(X,Y) for $\{G, f, w\}$ is one where $h = W(X = f, b = w)$ is either ① a nontriv elt of G with Relength$(h) <$ Relen(f) or ② $h \cdot A_f$ is closer to A_f than $w \cdot A_f$.

When $\{G, f, w\}$ has a killer word, it is not in $\mathcal{J}$.

Now suppose $[P] \in (\mathbb{I}\mathbb{C})^3$. A killer word for $[P]$ is one that works for all $\{G, f, w\} \in [P]$.

[GMT] Divided $\mathcal{P}$ into ≈ 1 billion $[P]$ and Found killer words for all but 7 tiny ones.



For each exception box, a few short words were found which are ≈ 1 in $PSL_2\mathbb{C}$, in fact must be 1 by Jørgensen. Showed there is a unique pt in box where relation holds. Used relations to build a genus 2 Heegaard decomp. of these manifolds...