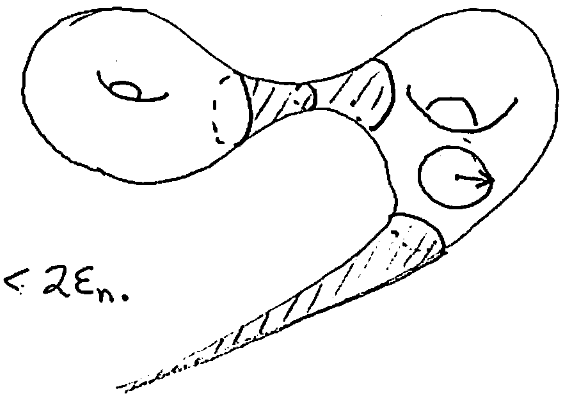
Lecture 14: Volumes of hyperbolic 3-manifolds

Margulis Lemma: $\exists \varepsilon_n > 0$ such that for all finite-volume hyp. M^n the set

$$M_{thin, \varepsilon_n} = \{ x \in M \mid inj_x M < \varepsilon \}$$



is a finite union of cusp nbhds and tubes about simple geod of len $< 2\varepsilon_n$.

Idea: Two isom that move a point $p \in \mathbb{H}^n$ a small amount either commute or generate an indiscrete group.

Quantitative version: The shorter the geod γ the larger its tube radius. So: very short geod has a tube that's almost a cusp.

[True for infinite vol M^n as well, with add'l cusp types.]

Thick-Thin Decomp. Fix some Margulis const ε_3.

Set $M_{thin} = M_{thin, \varepsilon_3}$ and $M_{thick} = M \setminus M_{thin}$.

Note: M_{thick} is compact with $\partial =$ tori. Get M from M_{thick} by Dehn filling and deleting some comps of ∂M_{thick}. [Q: Is the thick part itself hyp?]

Thm: Given V_0, there exist $X_1, \ldots, X_n$ so that every hyp M^3 with vol $\leq V_0$ has M_{thick} homeo to some X_i.

Idea: As $\mathrm{inj}_x M_{thick} \geq \varepsilon_3$ can triang M_{thick} by tetrahedra with edges $\geq \varepsilon_3/10$ each of which has volume $\geq \delta_3$. In part, can triang M_{thick} using $\leq V_0/\delta_3$ tets.

Cor: Given V_0, there exists a link $L_0 \subseteq S^3$ s.t. every hyp 3-mfld with vol $\leq V_0$ is obtained by doing Dehn surgery on some comps of L_0 and drilling out others.

Thm: If M^3 and a Dehn filling $M(\alpha)$ are both hyp, then $\mathrm{Vol}(M) > \mathrm{Vol}(M(\alpha))$.

Thm If M^3 is hyp and $\alpha_1, \alpha_2, \ldots$ are distinct slopes, then $\mathrm{vol}(M(\alpha_i)) \longrightarrow \mathrm{vol}(M)$. Also true for mult. cusps.

[In fact $M(\alpha_i) \longrightarrow M$ in the Gromov-Hausdorff sense]

Cor: Only finitely many hyp M^3 of the same volume.

Pf: Suppose $\{M_i\}_{i=1}^{\infty}$ are dist. with same vol. Passing to a subseq, can ensure ⓐ all $(M_i)_{thick}$ are homeo to X. If ∂X has k comps, have $M_i = X(\alpha_1^i, \alpha_2^i, \ldots, \alpha_k^i)$

ⓑ For each l, can assume either all α_l^i are equal or all are distinct. Reorder so $\alpha_m^i, \ldots, \alpha_k^i$ are const and set $Y = X(\_, \_, .., \alpha_m^i, .., \alpha_k^i)$.

Now all M_i are Dehn fill on hyp Y, so $\mathrm{vol}(M_i) < \mathrm{vol}(Y)$. As $\mathrm{vol}(M_i) \to \mathrm{vol}(Y)$, this contradicts that $\mathrm{vol}(M_i)$ is constant. ▨

Jørgensen-Thurston: The set of volumes of hyp M^3 is a well-ordered subset of $\mathbb{R}$ of order type Ω^{Ω}.

Every subset has a smallest elt.

Note: In all other dims, the set of volumes of hyp. n-mflds is discrete. [Q: What happens for $n = 2$?]

Weeks manifold:
smallest of
them all

(5,1)  (5,2)

 (5,1)

Vol3

First limit
point

$S^3 \setminus$ 

+ sibling

Second limit
point

$S^3 \setminus$ 

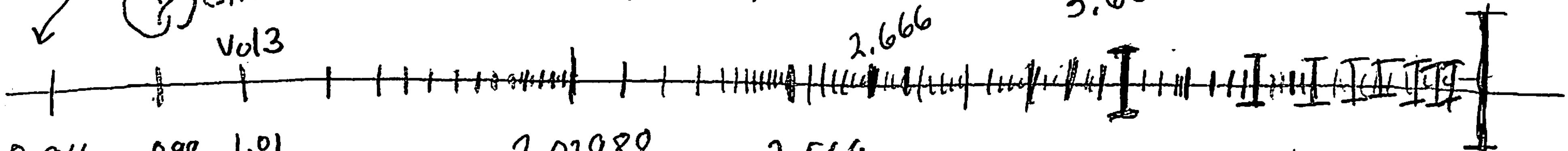
+

(-2,3,7)
pretzel

2.666

"Ω^2"
=
Vol of reg
ideal oct
=
3.6638

"Ω^3"
=
5.333



0.94… 0.98 1.01 2.02988… 2.564

"1" "2"

2.02988… = 2·(vol reg. ideal tet) = "Ω"

Gabai, Meyerhoff, N. Thurston,
Milley, Yarmola, …

Hodgson-Weeks, Matveev-Fomenko
layed out the conj. picture.

First limit pt
of limit pts

$S^3 \setminus$ 

[Agol] (-2,3,8) pretzel.
Only one where proof
does not involve rig.
computer computations.

Conject.
first trip.
limit pt