Lecture 7: Foliating all closed 3-manifolds

Previously: a) Reeb solid tori

b) Mapping tori / fibering over S^1: $M_f = F \times [0,1] \big/ (x,1) \sim (f(x),0)$ ← surface with ∂

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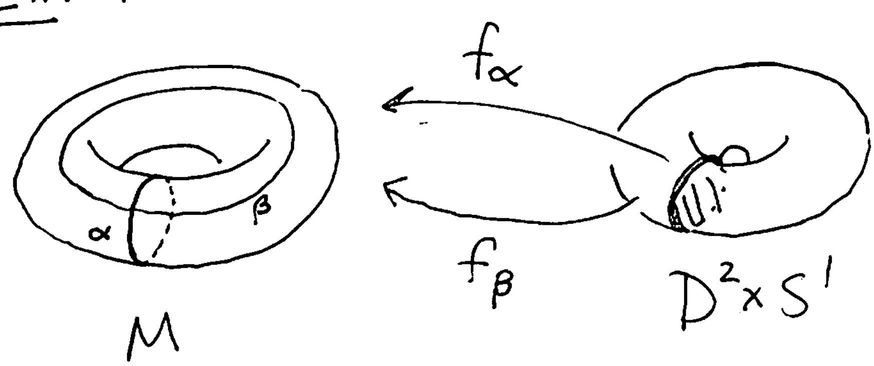
Dehn filling: Suppose $\partial M^3 = T^2$ and α an ess. simple closed curve on ∂M. Create a clsd $M(\alpha) = D^2 \times S^1 \cup_f M$ where $f : \partial(D^2 \times S^1) \to \partial M$ takes $\partial D^2 \times pt$ to α.

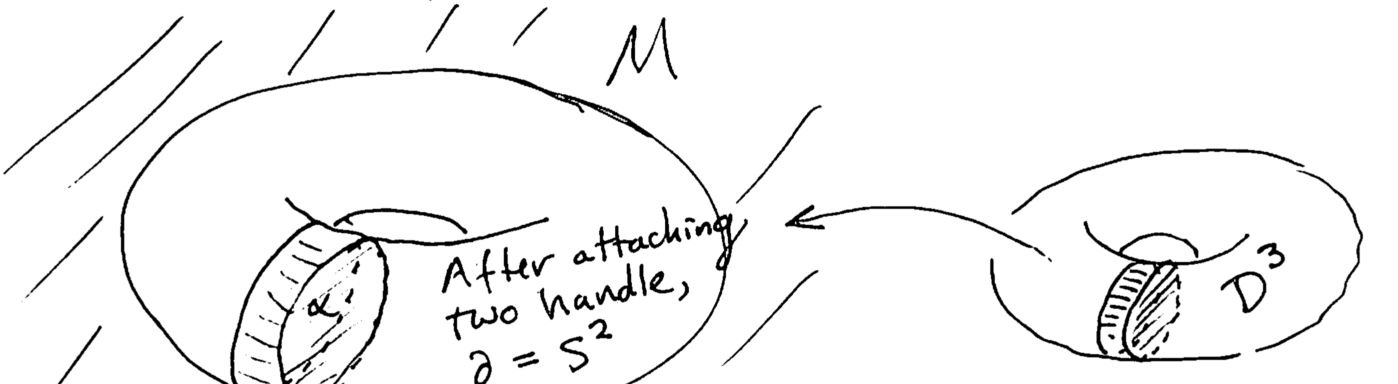
Ex: $M = D^2 \times S^1$ $\quad \alpha = \partial D^2 \times \{pt\}$ $\quad \beta = \{pt\} \times S^1$



$M(\alpha) = S^2 \times S^1$

$M(\beta) = S^3$

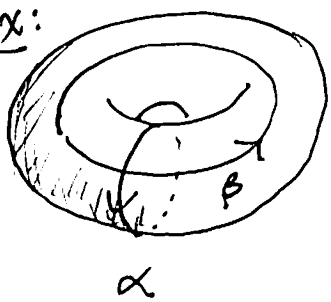
Note: $M(\alpha)$ is unique up to diffeo even though f is not (even up to isotopy):



Fact: Any diff of S^2 extends over D^3.

Note: Choice of α cor. to primative elt in $H_1(\partial M; \mathbb{Z})$ modulo sign.

Ex:



$M = D^2 \times S^1$

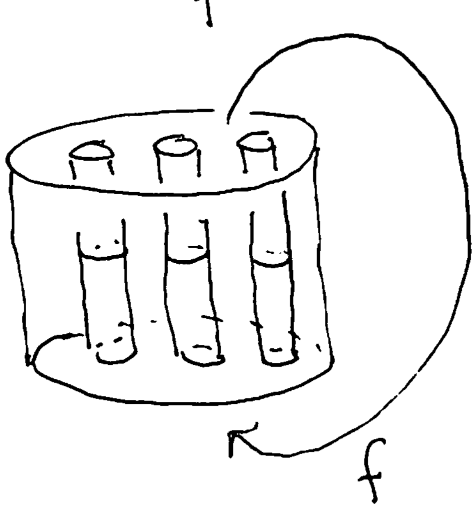
$$M(\alpha + \beta) = S^3$$
$$M(\alpha + 2\beta) = \mathbb{RP}^3$$
$$M(a\alpha + b\beta) = L(b, \pm a)$$

Note: $\pi_1 M(\alpha) = \pi_1 M / \langle\langle \alpha \rangle\rangle$

[Thurston] $M = S^3 \setminus \mathring{N}($  $)$. Then all but 10 Dehn fillings on M are hyperbolic.

[Same for any nontorus / nonsatellite knot]

Thm: Every closed orient N^3 is a Dehn filling on M_f where F is a cpt surface with boundary.

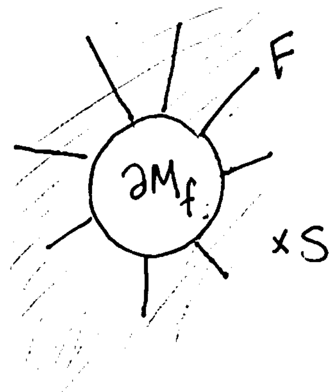


$\partial M_f =$ two tori.

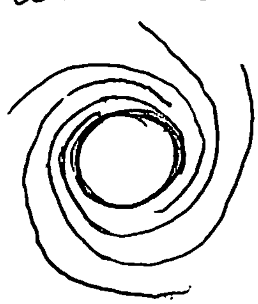
<u>Cor</u>: Every clsd orient N^3 has a co-orient $\mathcal{F}$.

<u>Pf</u>: Have $N = M_f(\alpha_1, \dots, \alpha_k)$ for some $f : F^2 \circlearrowleft \cong$.

Now M_f is fol by copies of F, transverse to ∂M_f.



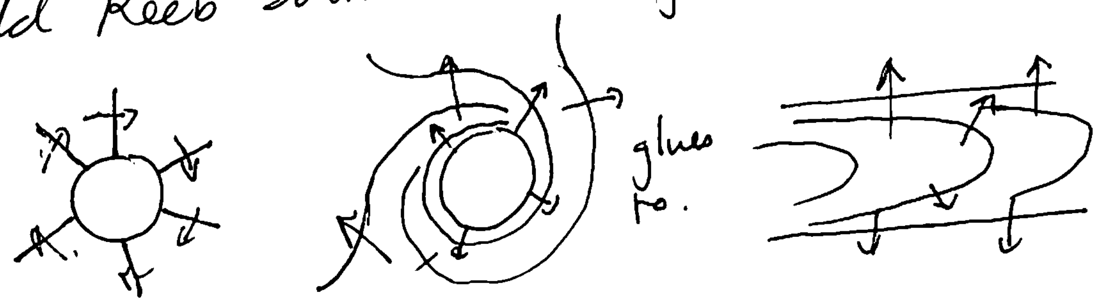
Now spin along M_f to get a new fol where ∂M_f is a tangential leaf:



[E.g. spinning product fol on $D^2 \times S^1$ produces the Reeb fol]

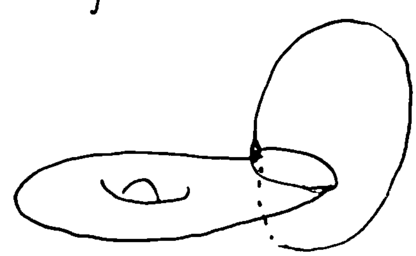
Now add Reeb solid tori to get a fol of N. ▨

<u>Co-orient</u>:

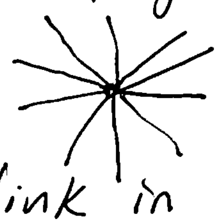


If $f|_{\partial F} = id_{\partial F}$ then there is a default choice for α_i, namely $\{pt \text{ in } \partial F\} \times S^1$

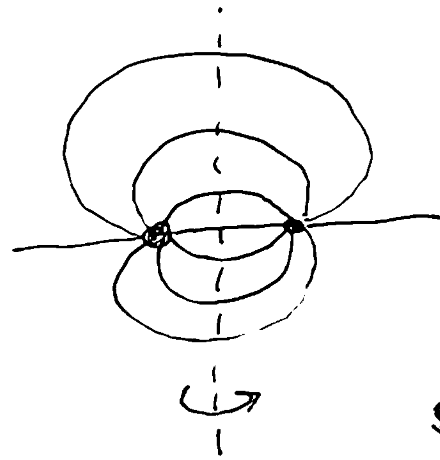
Such a Dehn filling is an <u>open book decomposition</u>.



Same as taking M_f alone and collapsing each {pt in ∂M} $\times S^1$ to a pt.

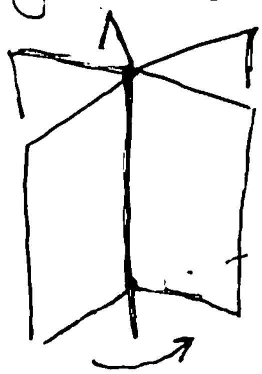


The comps of ∂F turn into a link in N (the binding) and the various copies of F are the pages.



Ex: $F = D^2$, $f = id$, gives an open book decomp of S^3

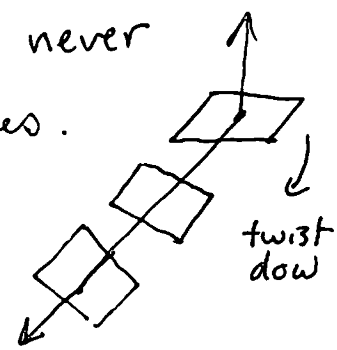
Put binding through ∞



Contact str:

$$\xi = \ker(dz + r^2 d\theta)$$

Transverse to binding, never tangent to the pages.



Thm [Thurston-Winkelnkemper 1975]
Every open book decomp. supports a contact str "like this".

Cor: Every closed orient M^3 has a co-orient contact structure.

[Giroux 2000] M^3 clsd orient.

$$\left\{\begin{matrix}\text{coorient contact}\\ \text{structures / isotopy}\end{matrix}\right\} \xleftrightarrow{\text{biject.}} \left\{\begin{matrix}\text{open book decomp}\\ \text{up to positive stabilization}\end{matrix}\right\}$$

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To prove thm on page 33, will first study other universal descriptions of M^3.

<u>Thm</u>: Any smooth M^n has a smooth triangulation. In particular, M is homeo to a simplicial complex.

[Freedman 80s; Manolescu 10s] For each $n \geq 4$ there exist compact topological n-mflds not homeo to any simp. cplx!

If time remains, talk about Heegaard splittings.