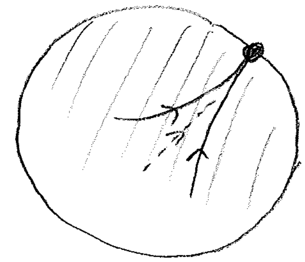
Lecture 15: More on Thurston's Univ. Circle



Recall: $\mathbb{H}^2 = \{|z|<1 \mid z \in \mathbb{C}\}$

$S^\infty_1 =$ equiv classes of geod rays

[Candel] If $\mathcal{F}$ is a taut fol of an irred atoroidal orient M^3, then $\exists$ metric on M s.t. every leaf has const curve -1.

[Q: What goes wrong for toroidal M?]

Cor: $\widetilde{\mathcal{F}}$ of $\widetilde{M}$ is a fol by hyp planes.

Goal: Unify the $S^1_\infty(\widetilde{L})$ into one circle with a $\pi_1 M$ act.

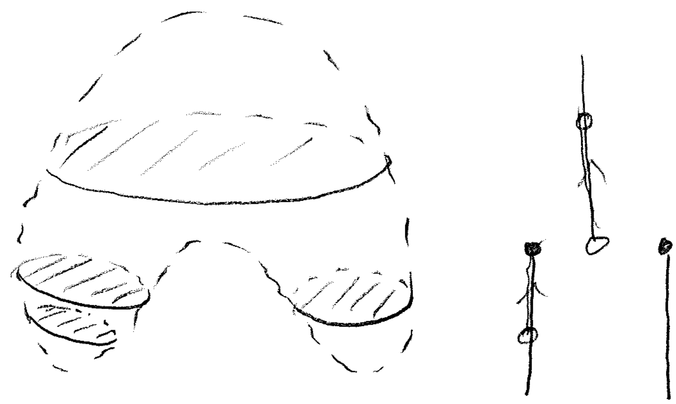
Note: Intrinsic geom of $\widetilde{L}$ is $\mathbb{H}^2$, extrinsic geom can be crazy. For M_f, each $\widetilde{F} \subseteq \mathbb{H}^3$ limits to all of S^2_∞. Get $\pi_1 F$ equiv. space-filling curve

$S^1_\infty(\widetilde{F}) \longrightarrow S^2_\infty(\widetilde{M})$ [Cannon-Thurston]

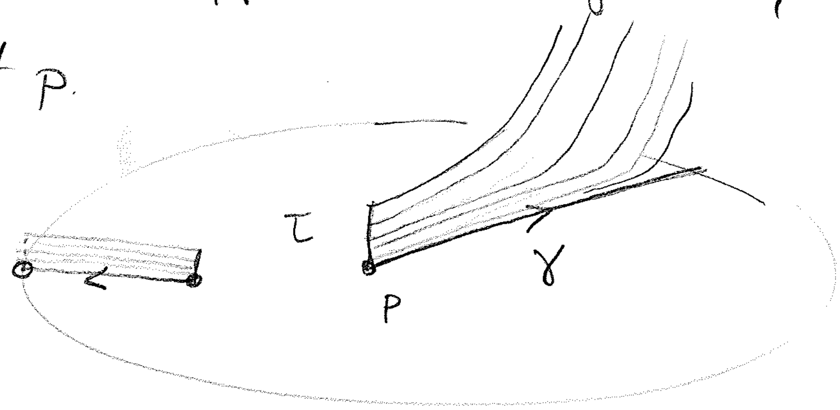
Two issues:

1) Non-Hausdorff leaf space.

Some leaves are "incomparable": no trans. meets both.



2) Even comparable leaves may not be uniformly close. Suppose γ is a geod ray in $\tilde{L}$ starting at p.



Can we find a transv. τ at p so that the hol of τ is defined along the whole of γ?

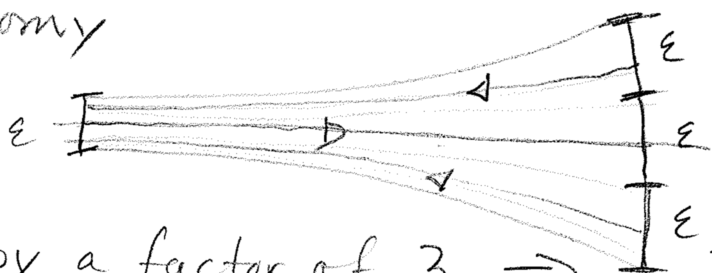
Marker: $I \times \mathbb{R}^+ \to \tilde{M}$ where

a) each $t \times \mathbb{R}^+$ is a geod ray in a leaf $\tilde{L}$.

b) each $I \times s$ is a transv to $\mathcal{F}$ of len $\le \varepsilon_0$.

<u>Leaf Pocket Thm</u>: For every $\tilde{L}$ of $\tilde{\mathcal{F}}$, the end pts of markers are dense in $S^1_\infty(\tilde{L})$

[Thurston proved this using harmonic measures on the leaves to study the behavior of random walks on a fixed leaf. Cartoon of idea: A transv. where the holonomy



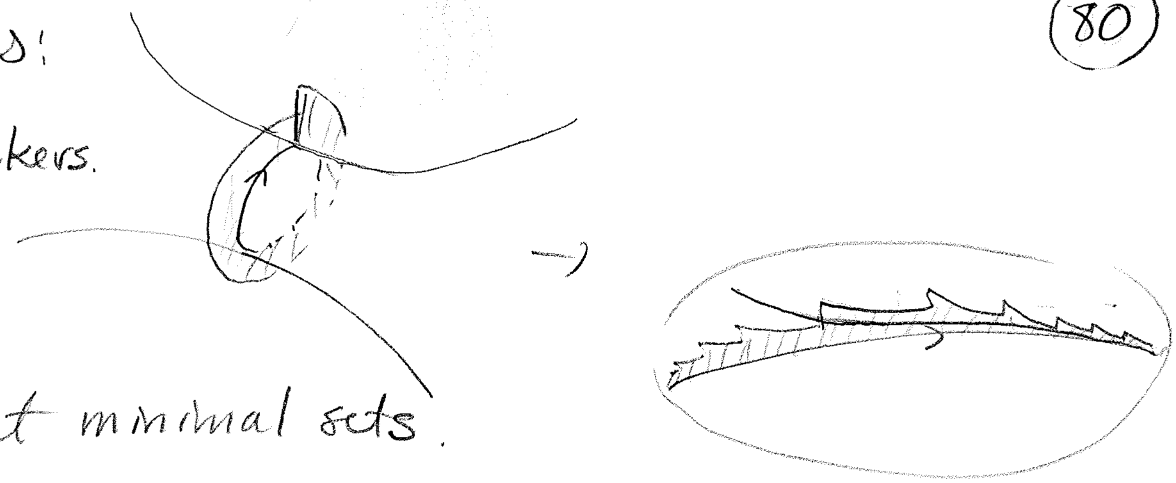
expands by a factor of 3 $\Rightarrow$ $\exists$ 3 transv where the holonomy contracts by $1/3$. Thus contract. holonomy is generic. Will instead follow [CD 2003]]

<u>Idea</u>: 1) Some leaf L of $\mathcal{F}$ has $\pi_1 L \neq 1$. Otherwise, no holonomy $\underset{\text{[Plante]}}{\Longrightarrow}$ leaves of $\mathcal{F}$ have poly. area growth.

This is a contradiction as each leaf is $\mathbb{H}^2$ in this scenario.

2) Sawblades:

Lead to markers.



3) Argue about minimal sets.

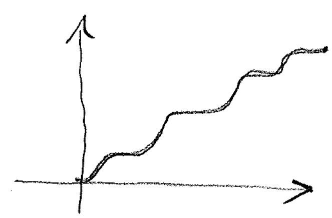
(e.g. suppose every leaf is dense.)

A <u>universal circle</u> S^1_{univ} for $\mathcal{F}$ is

a) An action of $\pi_1 M$ on S^1_{univ} $\left(\pi_1 M \xrightarrow[\rho_{univ}]{} \mathrm{Homeo}(S^1_{univ})\right)$

b) For every leaf $\tilde{L}$ a monotone map

$$\phi_{\tilde{L}} : S^1_{univ} \longrightarrow S^1_\infty(\tilde{L})$$

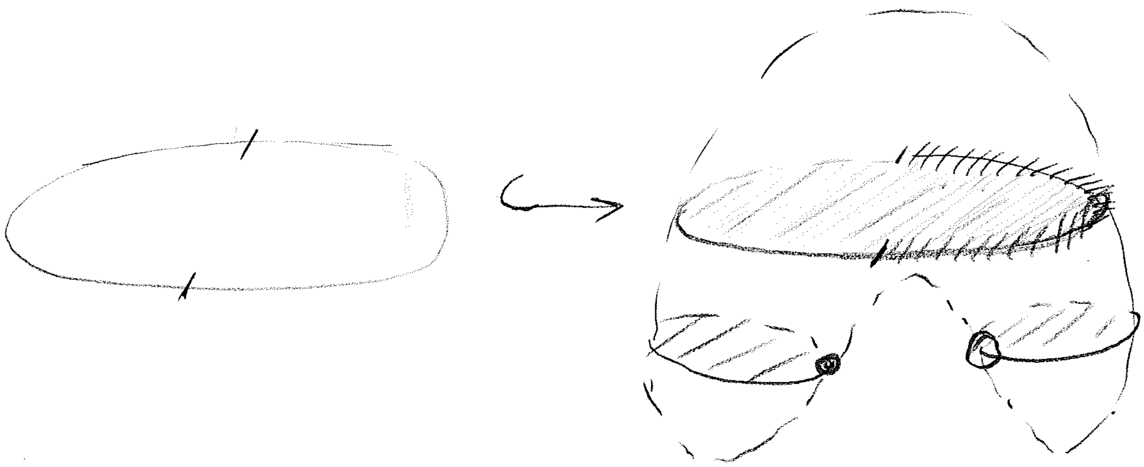


where $\forall\, \tilde{L}$ and $\alpha \in \pi_1 M$ one has

$$\begin{tikzcd}
S^1_{univ} \arrow[r, "\rho_{univ}(\alpha)"] \arrow[d, "\phi_{\tilde{L}}"'] \arrow[dr, phantom, "\circlearrowright"] & S^1_{univ} \arrow[d, "\phi_{\alpha(\tilde{L})}"] \\
S^1_\infty(\tilde{L}) \arrow[r, "\alpha"'] & S^1_\infty(\alpha(\tilde{L}))
\end{tikzcd}$$

Thm: $\exists$ taut fol of irred orient clsd M^3. Then $\mathcal{F}$ has a univ. circle. Moreover, the action of $\pi_1 M$ on S^1_{univ} is faithful $\Longrightarrow$ $\pi_1 M$ is a subgp of $\mathrm{Homeo}(S^1)$.

Idea: Use markers to take an inverse limit of the actions on the individual $S^1_\infty(\tilde{L})$.



For more, see Calegari's book or [Calegari–D 2003].