Lecture 13: Univ. cover of taut fol.

[Roussarie-Thurston] Suppose $\mathcal{F}$ is a taut fol of clsd M^3 and S an immersed incomp surface in M. Then S is homotopic to either

a) a leaf of $\mathcal{F}$

b) an immersed surface S' with $S' \cap \mathcal{F}$ having only saddle tangencies.

Pf: Same as showing M is irreducible. ▨

[Palmeira] If $\mathcal{F}$ is a taut fol of clsd M^3 with M not covered by $S^2 \times S^1$, then $\widetilde{M}_{\text{univ}} = \mathbb{R}^3$ and $\widetilde{\mathcal{F}} = \mathcal{G} \times \mathbb{R}$ where $\mathcal{G}$ is a fol of $\mathbb{R}^2$ by lines.

Context: There exist non-cpt contractible W^3 that are not $\mathbb{R}^3$ [Whitehead mfld]. By Geometrization, if M^3 is irred with $|\pi_1 M| = \infty$, then $\widetilde{M}$ is $\mathbb{R}^3$.

Pf. See [Foliations II, Ap D.]. So far, we know leaves of $\widetilde{\mathcal{F}}$ are prop. emb. planes, and turn now to one of the key ideas in the proof. ▨

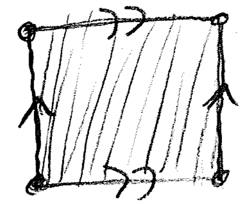
Leaf space: $\mathcal{F}$ fol of M the leaf space

$\mathcal{L} = M/\mathcal{F} = M/_{x \sim y}$ if x,y in the same leaf.

with the quotient top.

Ex: For $M = F \times S^1$, $\mathcal{L} = S^1$. Same for M_f.

Ex: $M = T^2$, $\mathcal{F}$ fol by line of irrat'l slope.



[Q: Is this taut?] Then $\mathcal{L}$ has

only two open sets: $\mathcal{L}$ and $\emptyset$.

Pf: $A \subseteq \mathcal{L}$ is clsd $\iff$ preimage under $M \to \mathcal{L}$ is clsd.

But any premage is a union of leaves and every leaf is dense. ▨

Note: $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ has $\widetilde{\mathcal{L}} = \mathbb{R}$.

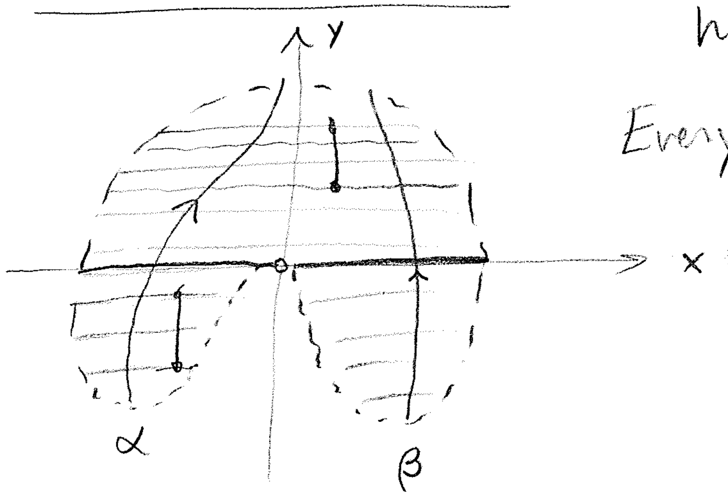
Thm: Suppose $\mathcal{F}$ is a taut fol of a clsd M^3.

Then the leaf space $\widetilde{\mathcal{L}}$ of $\widetilde{\mathcal{F}}$ is simply conn,

locally homeo to $\mathbb{R}$, and 2^{nd} countable.

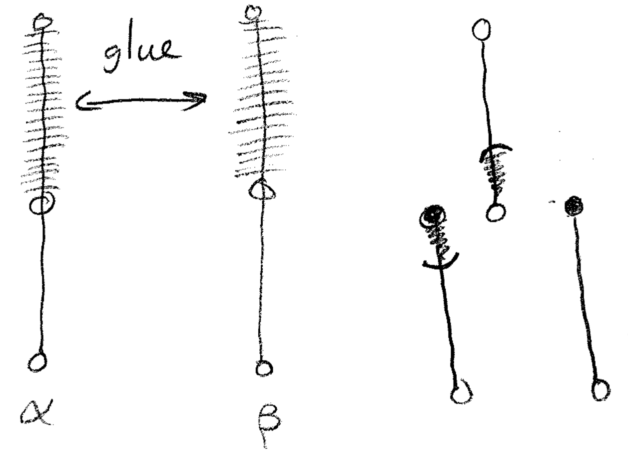
[Poss. non Hausdorff 1-manifold].

Consider $\widetilde{\mathcal{F}}$ of $\mathbb{R}^2$



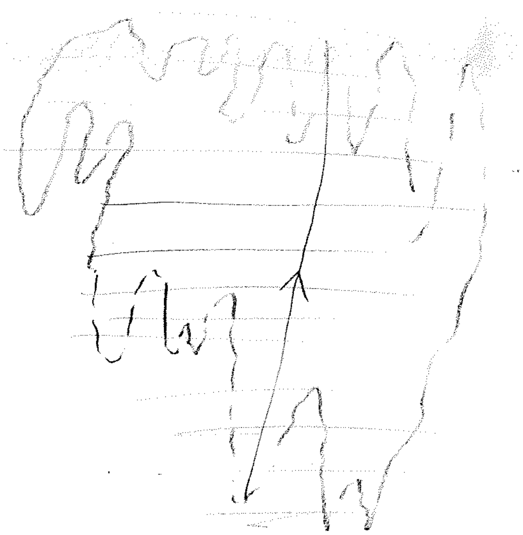
Transversals map homeomorphically into $\widetilde{\mathcal{L}}$.

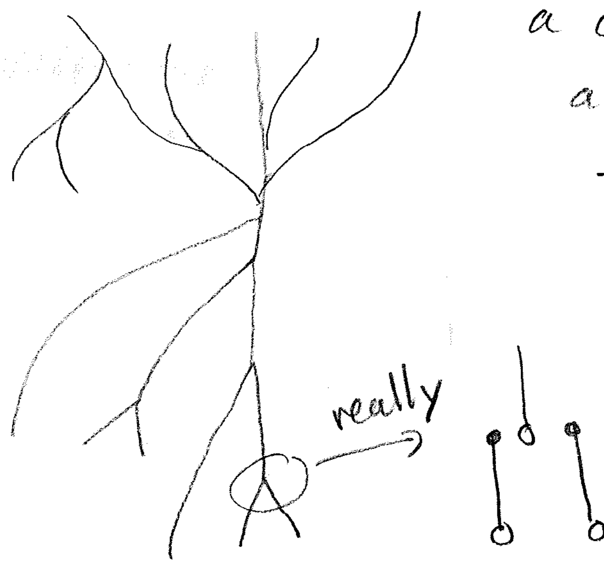
Every leaf meets α or β, so $\widetilde{\mathcal{L}}$ is



Not Hausdorff: can't separate the two leaves in the x-axis.

For a more realistic picture, "fractalize" since have a cocompact action of $\pi_1 M$.





Pf of Thm: Let τ be any open transv arc to $\widetilde{\mathcal{F}}$

Then $\tau \longrightarrow \widetilde{\mathcal{L}}$ is 1-1 since if not some leaf $\widetilde{L}$ meets τ in at least 2 pts, allowing use to construct a clsd transv to $\tilde{\mathcal{F}}$, a contradiction.

[As leads to clsd transv to $\mathcal{F}$ which has ∞-order in $\pi_1 M$.]

The map $\mathring{\tau} \longrightarrow \mathcal{L}$ is homeo onto its image, which is open, as for any open transv α the set $\bigcup\{\tilde{L} \mid \tilde{L} \text{ meets } \alpha\}$ is open in $\tilde{M}$. So $\tilde{\mathcal{L}}$ is locally homeo to $\mathbb{R}$.

Simply connected: Let γ be a loop in $\tilde{\mathcal{L}}$. Can construct $\tilde{\gamma} \subseteq \tilde{M}$ s.t. $\tilde{\gamma} \longrightarrow \tilde{\mathcal{L}}$ is a homeo onto γ. [Idea: cover γ by finitely many charts coming from some particular trans τ_i. Can arrange for γ to be a seg of linear seg. in these charts, now lift to the τ_i, connect by seg in leaves, push into gen pos]. Then $\exists D \rightarrow \tilde{M}$ with $\partial D = \tilde{\gamma}$ whose image in $\tilde{\mathcal{L}}$ shows γ is null homotopic. ▨

[Palmeira] In this setting, $\tilde{\mathcal{F}}$ is determined by $\tilde{\mathcal{L}}$. Also, $\tilde{\mathcal{F}}$ can be realized by taking an open disk in $\mathbb{R}^2$, intersect with the fol by horiz. lines and crossing with $\mathbb{R}$.

Which manifolds have taut fol?

[Gabai 1987] Any clsd irred orient M^3 with $b_1 > 0$ has a taut fol.

[Roberts-Shareshian-Stein 2003] There exist clsd hyp. 3-mflds with no taut fol.

Pf showed that $\pi_1 M$ did not act on any simply conn. poss non Hausdorff 1-mfld without a global fixed pt.

[Calegari-Dunfield 2003] Same result using Thurston's Univ. circle.

[Kronheimer-Mrowka-Ozsváth-Szabo 2007] Many such mflds, including all hyp 2-fold branched covers over alt. links.