Lecture 12: More on taut foliations.

<u>Thm</u> Suppose $\mathcal{F}$ is a taut fol of a clsd orient M^3

If $\mathcal{F}$ is not the std fol on $S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$, then

1) M is irreducible.

2) Every leaf is incomp.

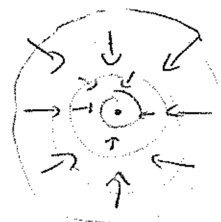
3) Every clsd trans. is $\neq 1$ in $\pi_1 M$.

New! 4) $\widetilde{M}$ is $\mathbb{R}^3$ with $\widetilde{\mathcal{F}}$ the product $\mathcal{G} \times \mathbb{R}$ where $\mathcal{G}$ is a fol of $\mathbb{R}^2$ by lines. [Palmeira]

[Discuss diff approaches, references...]

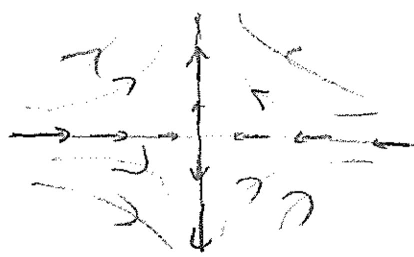
Euler char of surfaces: Generic zeros of a vector field on F

Center



(also minus this)

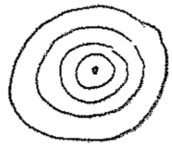
Saddle



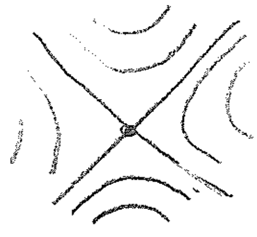
If X is generic on F and either trans or tangent to each comp of ∂F, then

$$\chi(F) = \sum_{\substack{\text{crit pt } p \\ \text{of } X}} I_p \qquad I_p = \begin{cases} +1 & \text{center} \\ -1 & \text{saddle.} \end{cases}$$

Suppose $\mathcal{F}$ is a singular fol of F with allowed sing.



center

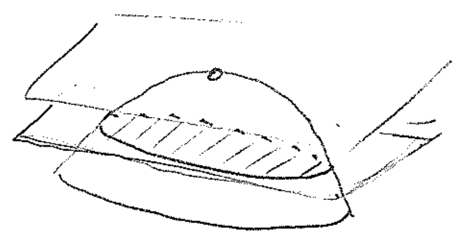


saddle

with each comp of ∂F either trans. or tang. to $\mathcal{F}$.

$$\chi(F) = \sum_{\text{sing } p} I_p .$$

Sketch of 1 via geom: By Sullivan, $\exists$ a metric on M where every leaf is a minimal surface (mean curve $= 0$). If M contains an ess. sphere, then $\exists$ a ess sphere of F of least area (which is embedded) [Sacks-Uhlenbeck; Meeks-Yau]. Assume F is generic w.r.t. $\mathcal{F}$. By the barrier princ, there will be no center sing of $F \cap \mathcal{F}$.

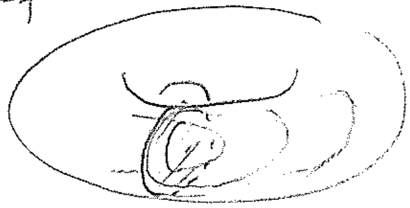


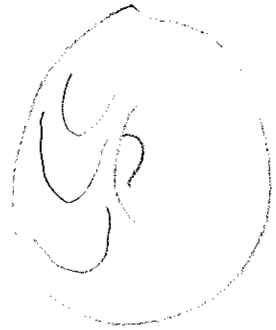
Thus $\chi(F) \leq 0$ a contradiction.

Main non-generic case is F is a leaf of $\mathcal{F}$ $\Rightarrow$ M covered by $S^2 \times S^1$.

Motivation: Reeb fol of S^3.

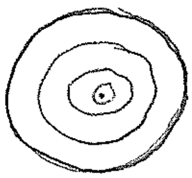
The torus leaf is comp,



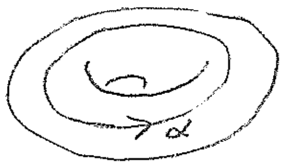


with

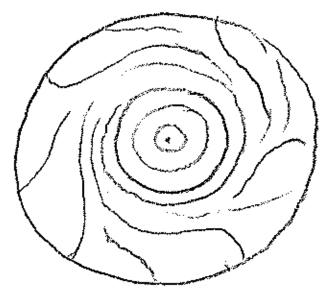
D

$\mathcal{F} \cap D =$  Each circle in $\mathring{D}$, bounds a disc in a leaf of $\mathcal{F}$.

The areas of these discs $\longrightarrow \infty$ They accumulate along the torus leaf, the only one that does not meet a clsd trans.

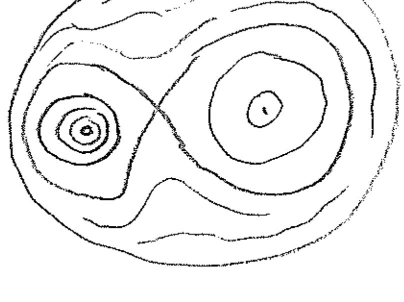
The core 

is a clsd transv but is 1 in $\pi_1 M$.

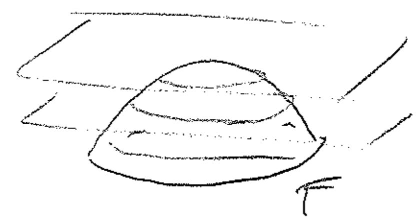
It bounds a disc D' which meets $\mathcal{F}$ like 

Idea in 2 and 3 is to study discs encoding that some $\gamma \subseteq L$ or trans α is 1 in $\pi_1 M$, by the fol $D \cap \mathcal{F}$. Since $\chi(D) = 1$, must be some center sing.

D



Near the center, each loop of $\mathcal{F} \cap D$ bounds a disc in a leaf L. This is an open cond, let $\mathcal{U} \subseteq D$ be all loops of $\mathcal{F} \cap D$ with this prop.



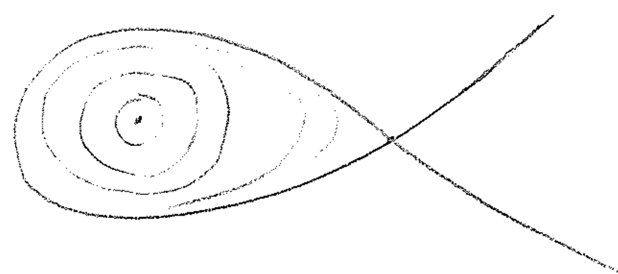
Various poss:

a) $\mathcal{U} = D \Longrightarrow \partial D$ bounds a disc in its L
so not a real compression.

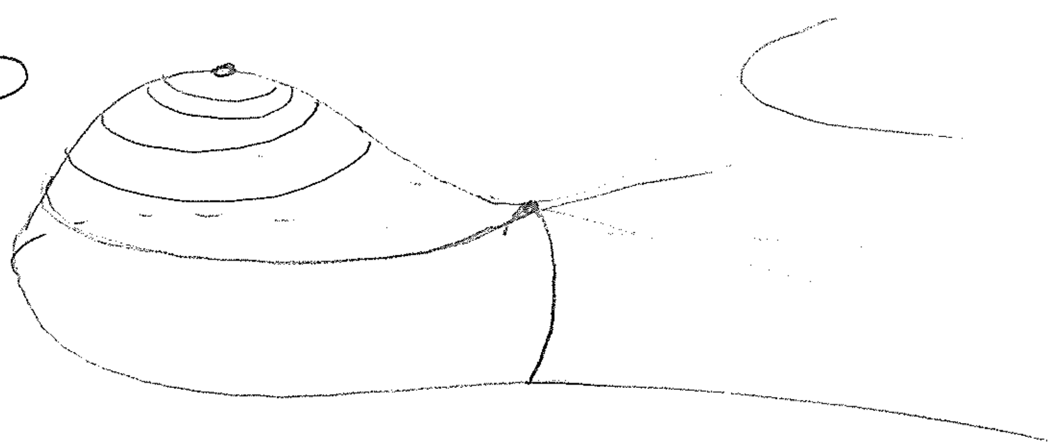
b) One component, ℓ, of $\partial \bar{\mathcal{U}}$ is a circle.
$\Longrightarrow$ "exploding disc" and a Reeb comp.

c) Something like

ℓ



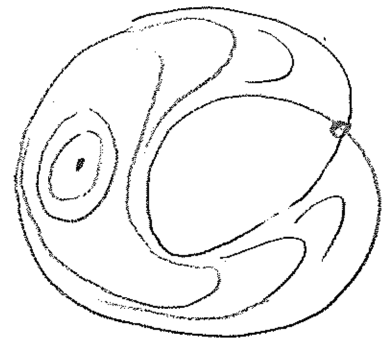
D



Now "cancel" this pair of sing.

[We're only req. that these discs be immersed, so don't have to worry that there is some other part of I in the "toe of the clown shoe".]

d) Variants of c)



Notes: Conclusions 2) and 3) only req $\mathcal{F}$ is Reebless, not taut. Novikov also showned a Reebless fol $\Rightarrow \pi_2 = 0$ [weaker than irred abs. the Poincaré conj]

[Roussarie-Thurston] Suppose $\mathcal{F}$ is taut and S an immersed incomp surface. Then S is homotopic to either

a) a leaf of $\mathcal{F}$.

b) intersect $\mathcal{F}$ only in saddle tangencies