Lecture 11: Taut foliations

Previously... 1) Every closed orient M^3 has a foliation.

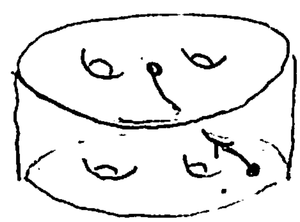
2) If L is a nonclosed leaf of $\mathcal{F}$, then $\exists$ a closed loop, transverse to $\mathcal{F}$, which meets L.

A closed transversal is a embedding $S^1 \hookrightarrow M$ which is transverse to $\mathcal{F}$.

Def: A foliation $\mathcal{F}$ of M^3 is taut if every leaf L has a closed transversal γ_L that meets it.

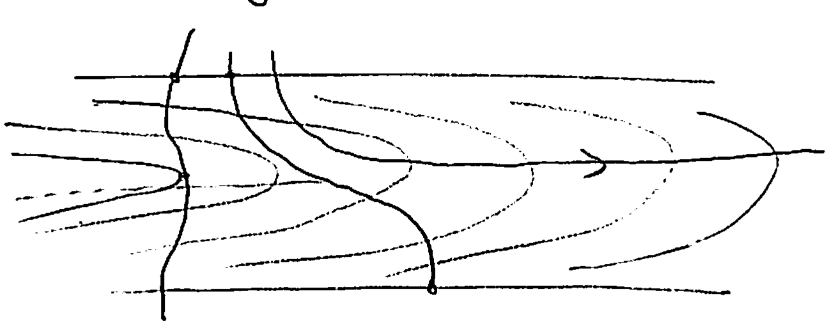
Ex: $F \times S^1$ $\qquad M_f$

$\gamma = \{pt\} \times S^1$



Ex: Any $\mathcal{F}$ of a cpt M^3 where no leaf is compact.

Non Ex: Any $\mathcal{F}$ with a Reeb component, that is containing a Reeb solid torus R. Issue: ∂R has no closed trans.

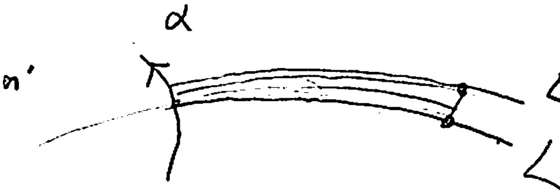


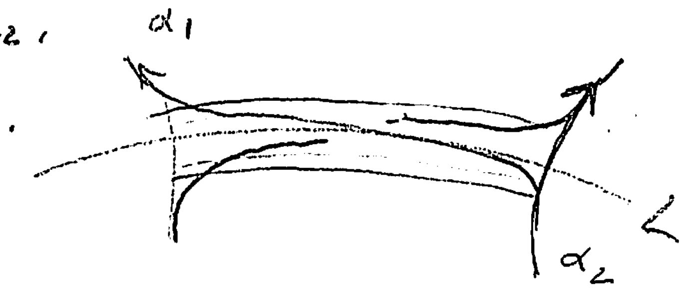
"Dead end component"

Pf: Look at unit vector field $\vec{n}$ pointing into R, consider $\langle \gamma'(t), \vec{n}_{\gamma(t)} \rangle$.

Prop: If M^3 is cpt and conn, $\mathcal{F}$ is taut $\iff$ $\exists$ a closed trans. γ that meets every leaf.

Pf. If α is a closed trans, $\cup\{L \subseteq \mathcal{F}$ meets $\alpha\} = U_\alpha$ is open. Cover M with $U_{\alpha_1}, \ldots, U_{\alpha_n}$. As M is conn, two overlap, say L is common to U_{α_1} and U_{α_2}. Create β with $U_\beta = U_{\alpha_1} \cup U_{\alpha_2}$. Repeat. ▨

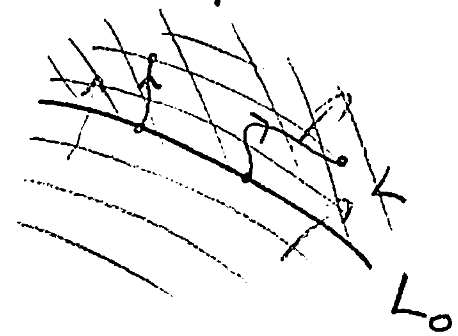




Thm: Suppose $\mathcal{F}$ is a co-orient fol of a clsd orient M^3. $\mathcal{F}$ is not taut $\iff$ there exist torus leaves $T_1, \ldots, T_K$ bounding a submfld V where the co-orient points into V everywhere.

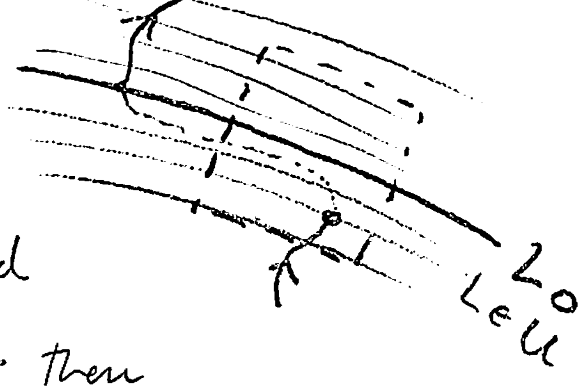
Ex of V: Reeb component, (Reeb annulus) $\times S^1 \subseteq T^2 \times I$

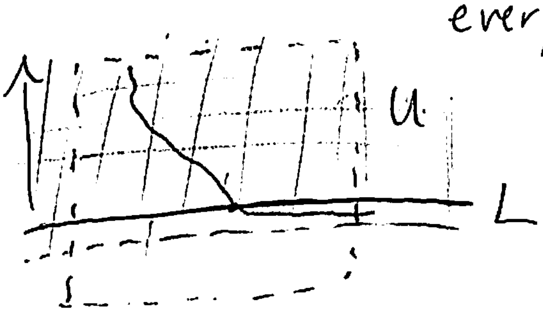
Pf idea: Let L_0 be a leaf not meeting any clsd transv; it must be cpt. Let U be the union of all $L \neq L_0$ s.t. $\exists$ a pos. trans from a pt in L_0 to a pt in L. Note U is open.



[Equiv, all L s.t. $\exists$ a pos trans from every pt in L_0 to every pt in L]

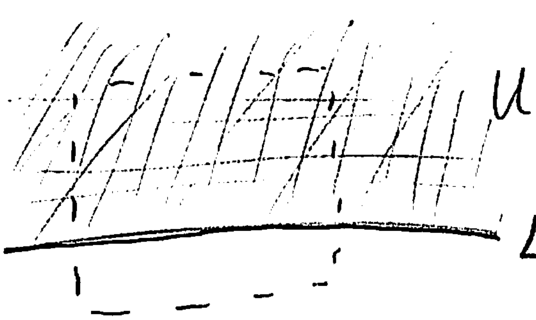
Also $\mathcal{U}$ does not approach L_0 from the negative side. So $V = \overline{\mathcal{U}}$ is not all of M. Now $V \setminus \mathcal{U}$ is a union of leaves and if L in $\mathcal{U}$ meets a fol chart then everything "above" L is in $\mathcal{U}$.





Thus $V \setminus \mathcal{U}$ meets a fol. chart in at most one plaque.

So $V \setminus \mathcal{U} = L_0 \cup L_1 \cup \cdots \cup L_n$ with all L_i cpt leaves with co-orient pointing in to V.



$L_1 \subseteq V \setminus \mathcal{U}$ No $L_i = S^2$ by Reeb stab. Inward nowhere vanishing vector field

$$\Rightarrow \chi(V) = 0 \Rightarrow \chi(\partial V) = 0 \Rightarrow \text{all } L_i = \text{T}^2. \;\blacksquare$$

Poincaré-Hopf Poincaré-Lefshetz: for M^3, $\chi(\partial M) = \frac{1}{2}\chi(M)$

<u>Thm</u>: Suppose $\mathcal{F}$ is a co-orient fol on a clsd orient M^3. Then $\mathcal{F}$ is taut $\Longleftrightarrow$ $\exists$ a transv. volume-preserving flow.

Note: A dead end comp makes such a flow impossible.

[Novikov-Rosenberg] Let $\mathcal{F}$ be a taut fol of a clsd orient M^3. If $M \neq S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$, then

1) M is irred.

2) each leaf L is incomp $(\pi_1 L \hookrightarrow \pi_1 M)$

3) every clsd trans is $\neq 1$ in $\pi_1 M$.

Cor: If M has a taut fol, $\pi_1 M$ is infinite. If $\tilde{\mathcal{F}}$ is the induced fol of $\tilde{M}_{univ}$, then every leaf of $\tilde{\mathcal{F}}$ is a prop. emb. plane. $(M \neq S^2 \times S^1, \mathbb{RP}^3 \# \mathbb{RP}^3)$

Pf of Cor: Let γ be a clsd trans to $\mathcal{F}$. Then γ^n can be pert. to a clsd trans $\Rightarrow \gamma^n \neq 1$ in $\pi_1 M$ for $n > 0$. So $|\pi_1 M| = \infty$. By (2) each leaf $\tilde{L}$ of $\tilde{\mathcal{F}}$ must be a plane. If $\tilde{L} \hookrightarrow \tilde{M}$ is not proper, it meets some prod chart in at least 2 plaques. $\Rightarrow \exists$ a clsd trans $\tilde{\gamma}$ to $\tilde{\mathcal{F}}$ meeting $\tilde{L}$. Then image γ of $\tilde{\gamma}$ in M is a clsd trans $\Rightarrow \gamma \neq 1$ in $\pi_1 M$ $\Rightarrow \gamma$ has inf order in $\pi_1 M \Rightarrow \gamma$ can't lift to $\tilde{\gamma}$ in $\tilde{M}_{univ}$.