Lecture 8: Foliating all clsd 3-mflds II

Goal:

Thm: Every clsd orient N^3 is a Dehn filling $M_f(\alpha_1, \ldots, \alpha_n)$ for some diffeo f of a surface F.

Cor: Every clsd orient N^3 has a co-orient $\mathcal{F}$.

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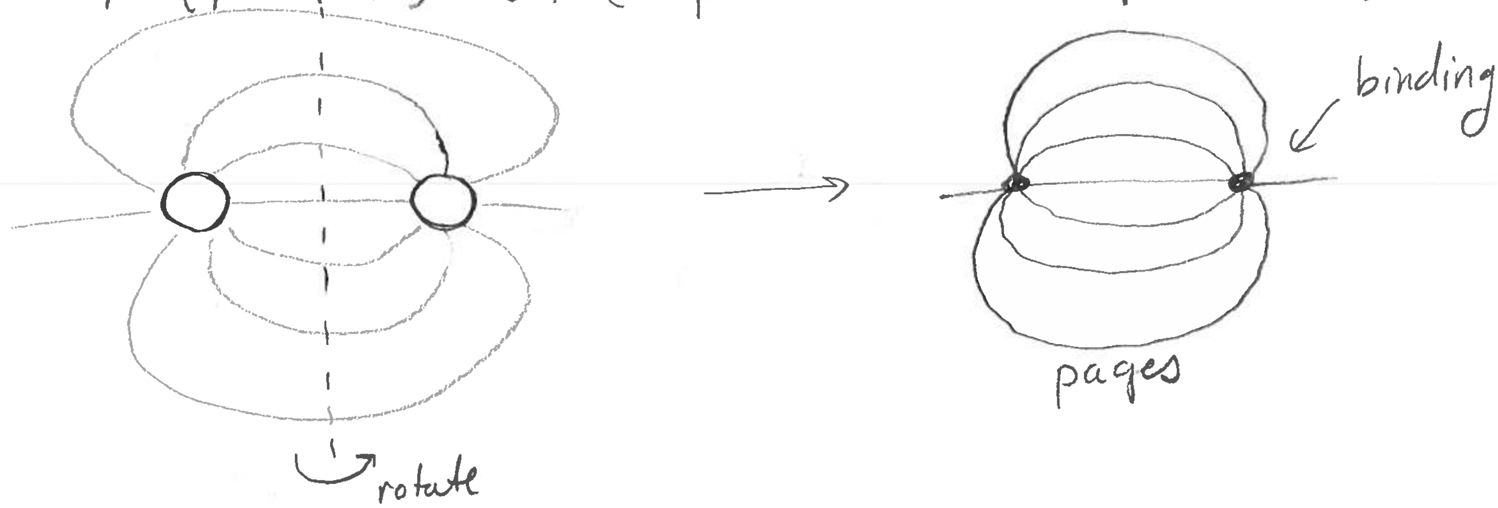
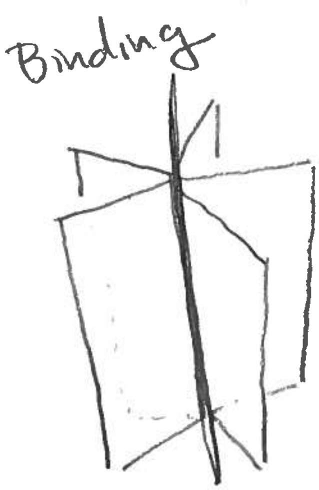
Addendum: Can require $f|_{\partial F} = id$ and each $\alpha_i = (\text{pt on } \partial F) \times S^1$. (Open book decomposition)

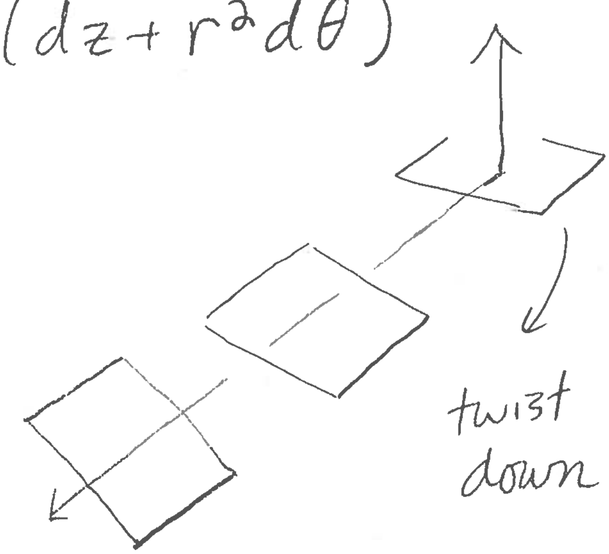




goes through ∞.

Contact str.

$$\xi = \ker(dz + r^2 d\theta)$$



<u>Thm</u> [Thurston–Winkelnkemper 1975]
Every open book decomp. supports a contact str "like this".

[Giroux 2000] M^3 clsd orient

$$\left\{\begin{array}{c}\text{co-orient contact}\\ \text{structures}/\text{isotopy}\end{array}\right\} \xleftrightarrow{\text{bijection}} \left\{\begin{array}{l}\text{open book decomp}\\ \text{up to pos. stab.}\end{array}\right\}$$

<u>Ref</u>: Etnyre's 2004 notes.

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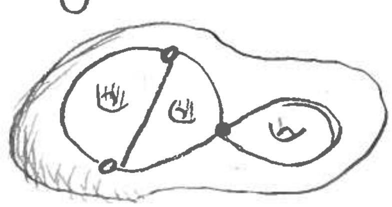
[Various "universal" descriptions of 3-mflds...]

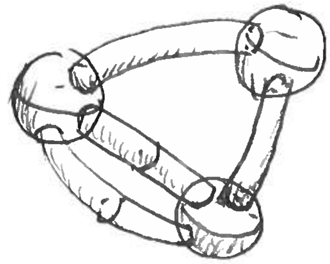
<u>Handlebody</u>: H_g is the cpt 3-manifold bounded by the standard genus g surface in $\mathbb{R}^3$:

Σ_3

Equiv, a handlebody is a clsd reg nbhd of a graph in $\mathbb{R}^3$



Equiv, a collection 0-handles with 1-handles attached.



$D^3 = 0$-handle

$D^2 \times D^1$ glued by $D^2 \times \partial D^1 = 1$-handle

<u>Heegaard splitting</u>: A surface Σ in M^3 dividing it into 2 handle bodies. Equiv, $M = H_g \cup_f H_g$ for some diffeo f of ∂H_g.

<u>Ex</u>: $g=0$, $H_0 =$  $\Rightarrow M = S^3$

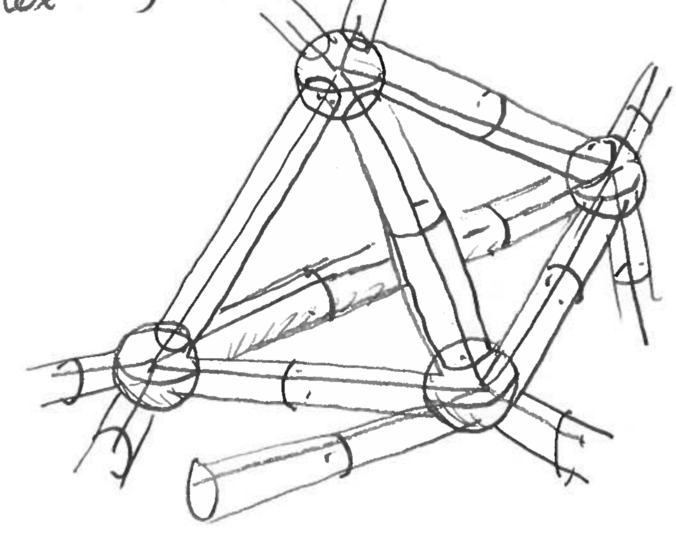
$g=1$, S^3, $L(p,q)$, $S^2 \times S^1$.

[Waldhausen] S^3 has a unique Heegaard splitting of genus g for each $g \geq 0$.

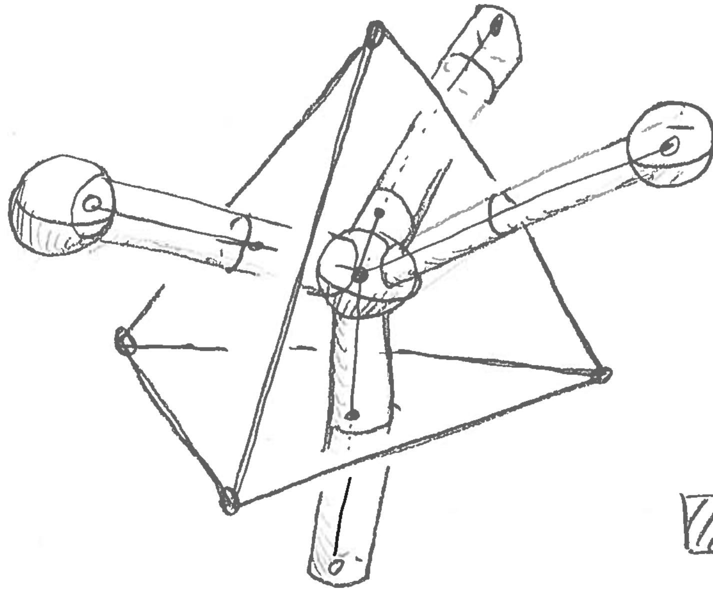
<u>Thm</u>: Every csld orient M^3 has a Heegaard splitting.

<u>Pf</u>. As a smooth mfld, M has smooth triangulation, i.e. there is a homeo $f: \left(\begin{smallmatrix}\text{simplical} \\ \text{complex}\end{smallmatrix}\ J\right) \to M$, which is smooth on each simplex.

<u>Claim</u>: A reg nbhd of $J^{(1)}$ is a handlebody



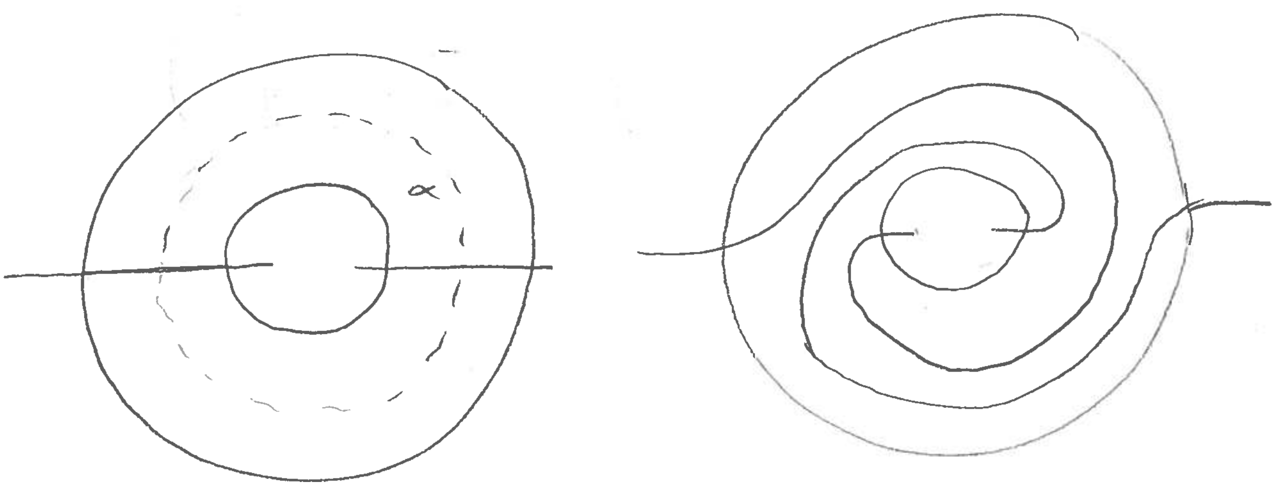
Claim: The complement is a reg nbhd of the dual 1-skeleton, hence also a handlebody



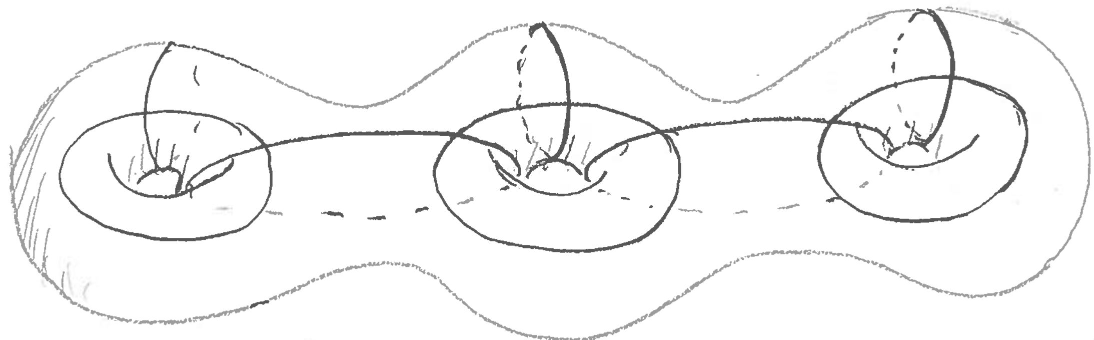
So M has a Heegaard splitting. $\square$

[Any two splittings are isotopic after stabilization.
To connect back to the original question, need to understand the poss. gluing maps.]

Suppose α is an ess simp. clsd curve on Σ.
The Dehn twist D_α is the diffeo of Σ that does this on an annulus nbhd of α and $=$ id elsewhere



[Lickorish] Any orient pres diff f of Σ_g is isotopic to a composition of Dehn twists along the following $3g-1$ curves



[Note: you many have to use each curve many times]

[Maher] A random Heegaard splitting is hyperbolic with prob $\longrightarrow 1$.

A <u>link</u> is an embedding $\amalg S^1 \hookrightarrow S^3$

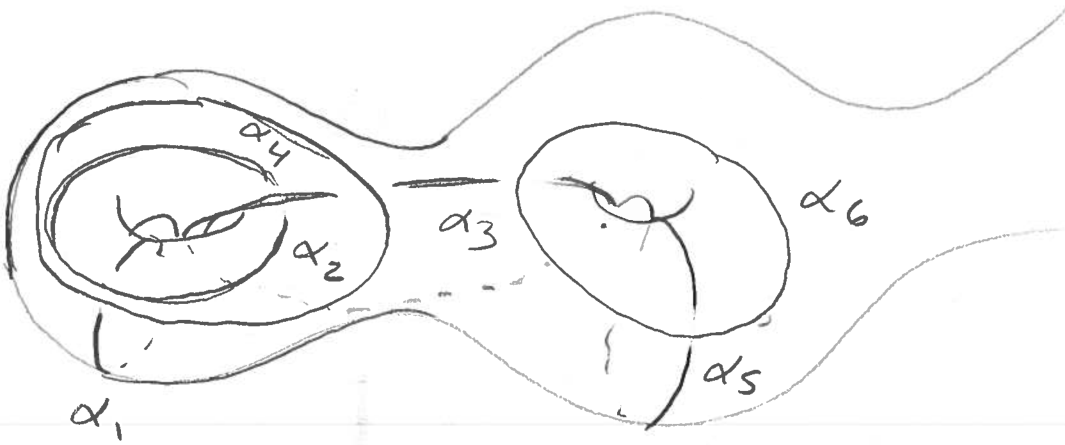
Its <u>exterior</u> is $S^3 \setminus \mathring{N}(L)$ which has $\#L$ tori boundary. A Dehn filling of the exterior is also called <u>Dehn surgery</u> on L.

<u>Thm</u>: Every clsd orient M^3 is Dehn surgery on a link L in S^3.

<u>Cor</u>: Every such $M^3 = \partial W$ for some smooth simply conn. W^4.

[Note: cor. holds for surfaces.]

Pf Idea: Implement f as a seq of Dehn twists along $\alpha_1, \alpha_2, \ldots, \alpha_n$ as Dehn surgeries along curves parallel to the α_i.



Formally: Let f_0 be s.t. $H_g \cup_{f_0} H_g$ is the std splitting of S^3.

Suppose $f_0^{-1} \circ f = D_{\alpha_n}^{\pm 1} \circ \cdots \circ D_{\alpha_1}^{\pm 1}$

Take a nbhd N of Σ_g in S^3 with $N = \Sigma_g \times (0, n+1)$

Set $L = \bigcup \alpha_i \times \{i\}$. Then approp. ± 1 Dehn surgery on the comps of L is exactly

$H_g \cup_f H_g$. $\blacksquare$