Lecture 23: Thurston norm and foliations.

Last time: M^3 clsd orient irred. For $c \in H_2(M; \mathbb{Z})$ set

$$\|c\|_{Th} = \min\left\{ -\chi(S) \;\middle|\; \begin{array}{l} S \text{ a nice surface (no comp is 0 in } H_2) \\ \text{that reps } c. \end{array} \right\}$$

A nice S is taut when $-\chi(S) = \|[S]\|_{Th}$.

Thm: Suppose F is a cpt leaf of a co-orient taut $\mathcal{F}$. Then F is taut.

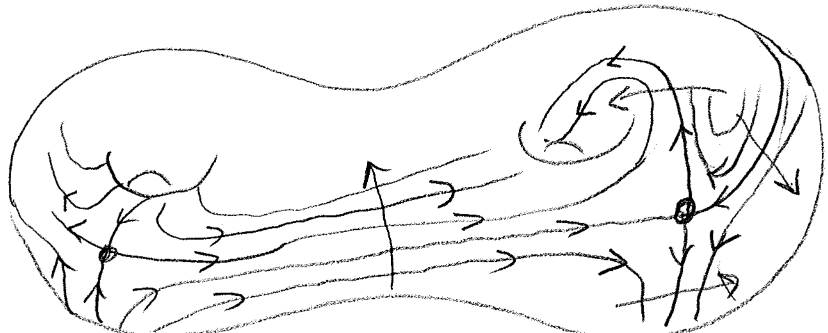
Can allow immersed!

Pf idea: Suppose S is a taut surface rep $[F]$. As S is incomp, can homotope S so transv to $\mathcal{F}$ except at finitely many saddle tang. Two kinds dep on whether T_pS and $T_p\mathcal{F}$ have the same orient. Set I_p = # orients agree, I_n = # disagree. Recall $I_p + I_n = -\chi(S)$.

Lemma: If $e(T\mathcal{F}) \in H^2(M)$ is the Euler class of $T\mathcal{F}$ then $e(T\mathcal{F})([S]) = e(T\mathcal{F}|_S)([S]) = I_n - I_p$.

Pf: See either text

but in a picture:



$\mathcal{F}$-co-orient

$\Rightarrow \mathcal{F} \cap S$ - co-orient

$\Rightarrow \mathcal{F} \cap S$ orient. $\Rightarrow$ gives section of $T\mathcal{F}|_S$ van. at tang.

Now

$$\|[S]\|_{Th} = -\chi(S) = I_p + I_n \geq I_p - I_n$$

$$= -e(T\mathcal{F})([S]) = -e(T\mathcal{F})([F])$$

$$= -e(T\mathcal{F}|_F = TF)([F]) = -\chi(F)$$

No comp of F sep, since each is meet by an orient clsd trans. So F is also taut. ▨

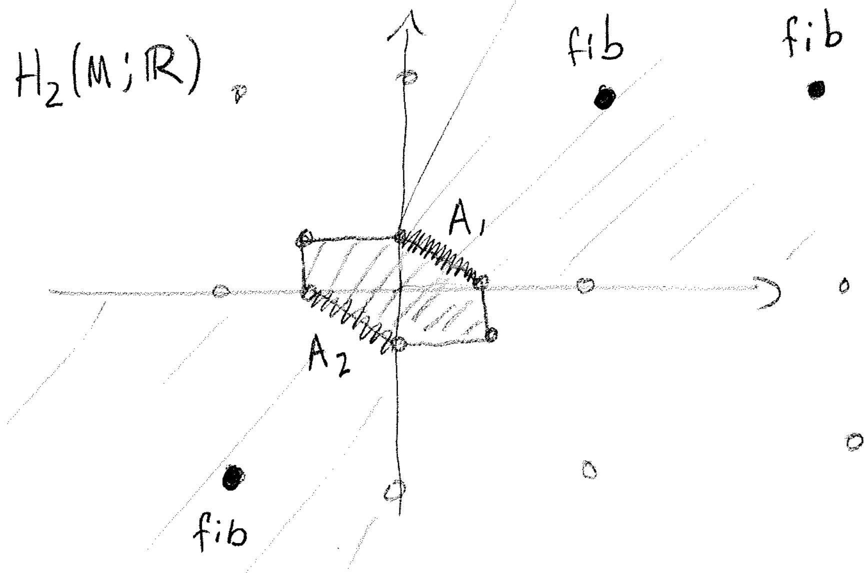
Cor: For M_f, each $F \times \{pt\}$ is taut.

Thm: M atoroidal so $B_T = \{ c \in H_2(M;\mathbb{R}) \mid \|c\|_{Th} \leq 1 \}$ is a finite rat'l polytope. There exist top dim'l faces $\{A_i\}$ of B_T such that

① $c \in H_2(M;\mathbb{Z})$ can be rep by a fiber in a fib over S^1 $\iff$ $c/\|c\|_{Th}$ in some $\mathrm{int}(A_i)$.

② $\phi \in H^1(M;\mathbb{R})$ can be rep by a nowhere vanishing 1-form $\iff$ $c = PD(\phi)$ has $c/\|c\|_{Th}$ in some $\mathrm{int}(A_i)$

Cor: If $b_1 > 1$, then some $c \neq 0$ in $H_2(M;\mathbb{Z})$ cannot be rep by a fiber.



Notes: 1) If $p: M \to S^1$ is a fib with fiber F then $PD(F)$ is rep by $p^*(d\theta)$ which is never 0.

2) The set of ϕ rep by nowhere 0 forms is open.

3) If $\phi \in H^1(M;\mathbb{Z})$ rep by nowhere 0 ω, then int. ω gives a fib $p: M \to S^1$.

[Gabai] If F is taut, $\exists$ a co-orient $\mathcal{F}$ with F as a leaf.

Cor: Suppose $F \subseteq M$ is taut. If $\tilde{M} \xrightarrow{p} M$ is a finite cover and $\tilde{F}$ a component of $p^{-1}(F)$, then $\tilde{F}$ is taut.

Pf: Let $\mathcal{F}$ be taut with F as a leaf.
Then $p^{-1}(\mathcal{F})$ is taut (why?) and has $\tilde{F}$ as a leaf.

[Won't be able to say anything about the proof of Gabai's thm in time remaining...]

Story time: M^3 clsd irred ator. If $H_2(M;\mathbb{Z}) \neq 0$ then $|\pi_1 M| = \infty$ since $c \neq 0$ can be rep by an incomp surface where every comp has genus ≥ 2. So M is hyp, metric is unique. [What about geom complex of c?] Mot. by $\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta$, on $H^1(M;\mathbb{R})$ set

$$\|\phi\|_{L^2}^2 = \inf \left\{ \int_M |\alpha|^2 \,\middle|\, \alpha \text{ reps } \phi \right\} = \langle \omega, \omega \rangle$$

where $\Delta \omega = 0$.

By P.D., have $\|\cdot\|_{Th}$ on $H^1(M;\mathbb{R})$ as well.

[Bergeron-Şengün-Venkatesh; Brock-D 2017]

$$\frac{\pi}{\sqrt{\mathrm{vol}(M)}} \|\phi\|_{Th} \leq \|\phi\|_{L^2} \leq \frac{10\pi}{\sqrt{\mathrm{inj}(M)}} \|\phi\|_{Th}$$

where $\mathrm{inj}(M) = \frac{1}{2} \min(\mathrm{len}(\text{closed geod}))$.

[If time remains, do least area norm.]