Lecture 16: L-space Conjecture

Previously: M clsd orient irred ator. with taut $\mathcal{F}$. Then $\pi_1 M$ acts on a simply conn "1-mfld" without a global fixed pt and $\pi_1 M$ acts faithfully on S^1.

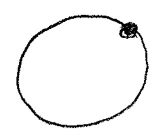
L-space Conjecture: M clsd orient irred. T.F.A.E.

1) M has a co-orient taut fol.

2) $\pi_1 M \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$.

3) Nontrivial (Heegaard/Seiberg-Witten) Floer homology ($\widehat{HF}_{red}(M) \neq 0$).

Note: $(2) \Rightarrow \pi_1 M$ acts on the "best" s.c. "1-mfld".
$\Rightarrow \pi_1 M$ acts faith. on S^1.



Only known implication is 1 $\Rightarrow$ 3 [KMOS; OS], which is used to show many mflds do not have taut fol by calculating $\widehat{HF}$. This connection goes through contact/symplectic geometry [Eliashberg-Thurston]

Holds for: 1) Seifert fibered spaces (fol. by circles)

2) Graph mflds (SFS glued along tori).

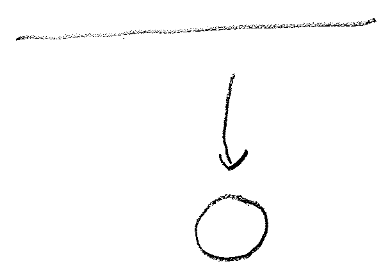
3) Whenever $b_1 > 0$. (has the fol)

4) 2-fold branched cover over an alternating link. (no fol)

5) At least 62.2% of the 307,301 small hyp M with $b_1 = 0$ I found on my computer. [G+T 2020]

6) Any M with a taut fol and $H_1(M;\mathbb{Z}) = 0$.

Pf: Action on S^1_{univ} lifts to an action on $\mathbb{R}$ as $H^2(M;\mathbb{Z}) = 0$.



The subconjecture ① ⟺ ③ holds for

1) All Dehn surgeries on alternating or Montesinos knots.

2) 99.8% of the small hyp M on my computer.

[Conj proposed by Boyer-Gordon-Watson, Oz-Sz, Juhász in about 2010. Discuss history.]

Most of these proofs are indirect, i.e. calculate the three props indep. and observe the lists match.

Inverse problem: Given an action of $\pi_1 M$ on $\mathbb{R}$, does this "come from" the leaf space of a taut fol?

For example, can we find an equivariant submersion $\tilde{M} \longrightarrow \mathbb{R}$?

[My intuition is no, but...]

Truth in advertising: Many fol $\mathcal{F}$ of M^3 one cares about are not C^∞ or even C^2. Which is why (2) is not about $\mathrm{Diff}^+(\mathbb{R})$.

[Calegari] Any top. fol of clsd M^3 is top isotopic to one where every leaf is C^∞-immersed. But $T\mathcal{F}$ may still be only a cont. subbundle of TM.

[Kazez-Roberts; Bowden] Clarified tautness, proved [ET] in the $C^{1,0}$ (and $C^{\infty,0}$) settings.

Issues: $T\mathcal{F}$ need not be uniquely integrable.

[ET] really uses C^2.

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A group G is <u>left-orderable</u> when it has a total order that is left inv: $a < b \Rightarrow ga < gb$

<u>Ex</u>: $(\mathbb{R}, +, <)$, F_n, B_n.

<u>Non Ex</u>: Any group with torsion: $1 < g \Rightarrow g < g^2 \Rightarrow 1 < g < \dots < g^n = 1$.

$SL_n\mathbb{Z}$ for $n \geq 3$.

<u>Thm</u>: A countable G is orderable $\Leftrightarrow G \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$.

<u>Idea for ($\Leftarrow$)</u>: Suppose G acts faith on $\mathbb{R}$ and $\mathrm{Stab}_{x_0} = \{1\}$ for some $x_0 \in \mathbb{R}$. Then define

$$g_1 < g_2 \Leftrightarrow g_1 x_0 < g_2 x_0.$$

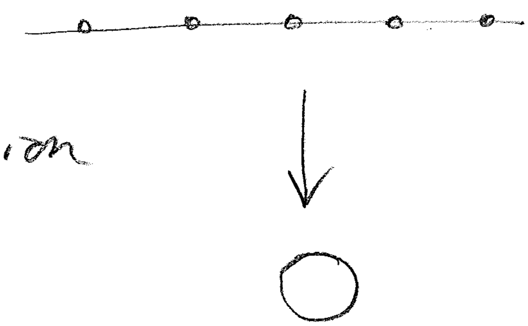
[In gen, use lexographical order on a count. dense set]

($\Rightarrow$) Use an order pres emb $G \to \mathbb{R}$ and extend.

Cf. Kaplansky's Conjecture: G torsion free $\Rightarrow \mathbb{C}G$ has no zero divisors holds for orderable G.

<u>Basic tool</u>: $\rho : \pi_1 M \longrightarrow PSL_2\mathbb{R} \longrightarrow \mathrm{Homeo}^+(S^1)$

$\widetilde{\mathrm{Homeo}^+(S^1)} \leq \mathrm{Homeo}^+(\mathbb{R})$ is the subgp of lifts of homeos of S^1, i.e. those commuting with action of $\mathbb{Z}$ by trans. Have



$$0 \to \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}^+(S^1)} \longrightarrow \mathrm{Homeo}^+(S^1) \to 1$$

Given $\rho : \pi_1 M \to \mathrm{Homeo}^+(S^1)$, can lift to $\widetilde{\mathrm{Homeo}^+(S^1)}$ iff $e(\rho) \in H^2(M; \mathbb{Z})$ is 0.

If $G = PSL_2\mathbb{R}$, let $\widetilde{G}$ be the preimage in $\widetilde{\mathrm{Homeo}^+(S^1)}$. Then $\widetilde{G}$ is the univ. covering group of $G \cong UT(\mathbb{H}^2) \simeq_{h.e.} S^1$.

[Discuss how this trick has been used...]