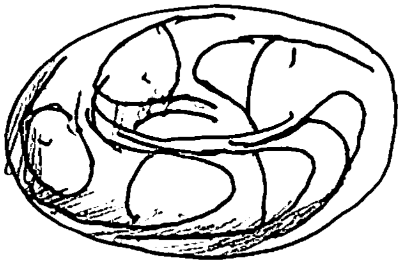
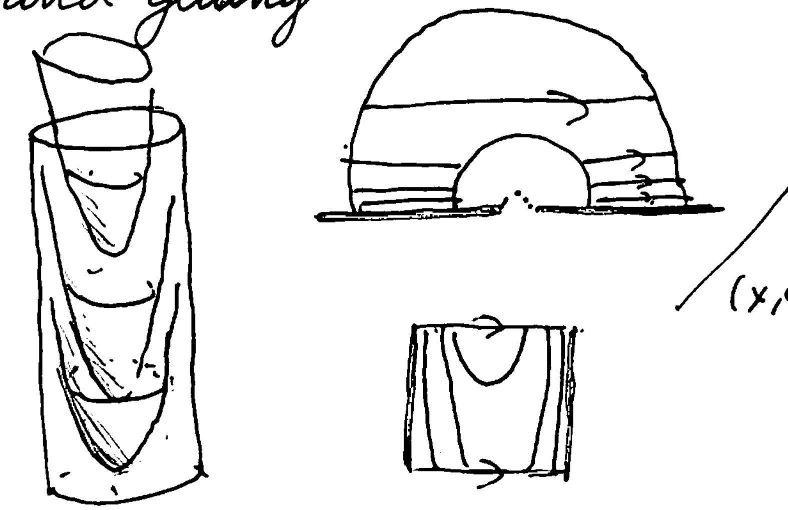
Lecture 4: Holonomy and gluing

Last time:



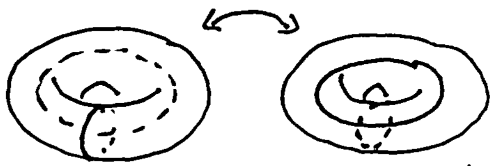
Reeb solid torus



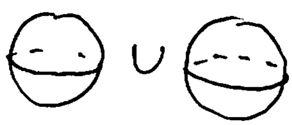
$(x,y) \to (2x,2y)$

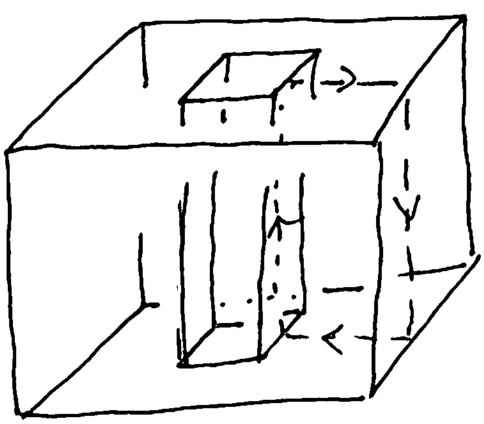
---

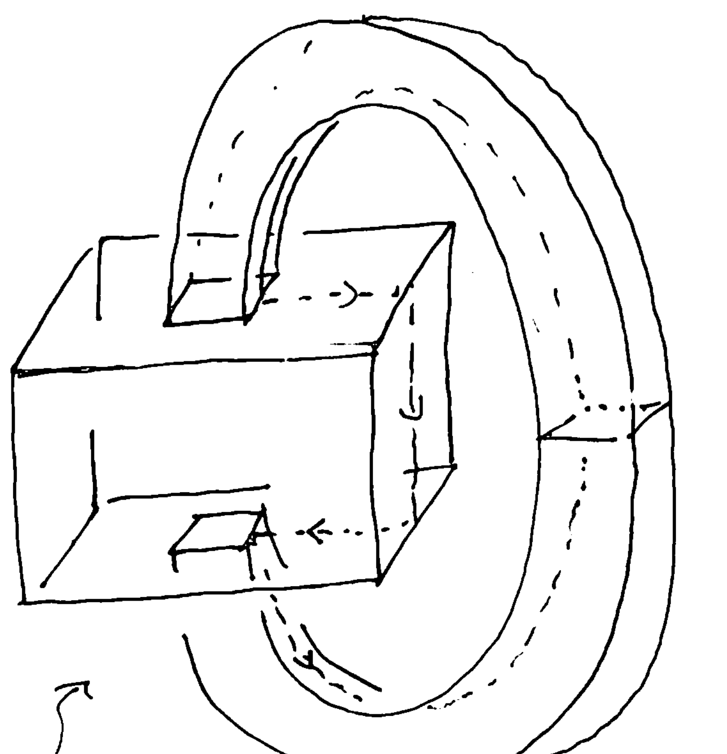
glue ∂.

Claim: $S^3 =$ 

$D^2 \times S^1$ $\quad$ $D^2 \times S^1$

Pf: $S^3 =$  $=$





remove column, glue to

Now compare which curves are glued to what.

[Reeb 1952] Foliate S^3 by making each of these into a Reeb solid torus.

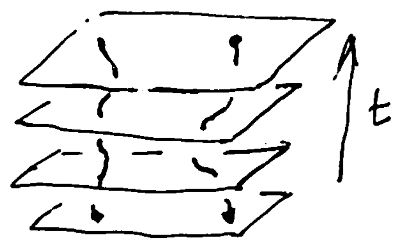
Q: Are we sure this is smooth? [In fact it's not without some modification.]

[Missing tool: Holonomy.]

//

$\mathcal{F}$ on M^3: a transversal $\tau: I \to M$ is a smooth arc transverse to $\mathcal{F}$ with $\tau(I) \subseteq$ fol. chart U.

Note: τ meets each plaque of U most once. If τ' is another transv in U with plaques P_0 and P_1 with $\tau(0), \tau'(0) \in P_0$ and $\tau(1), \tau'(1) \in P_1$, get a diffeo from τ to τ'.



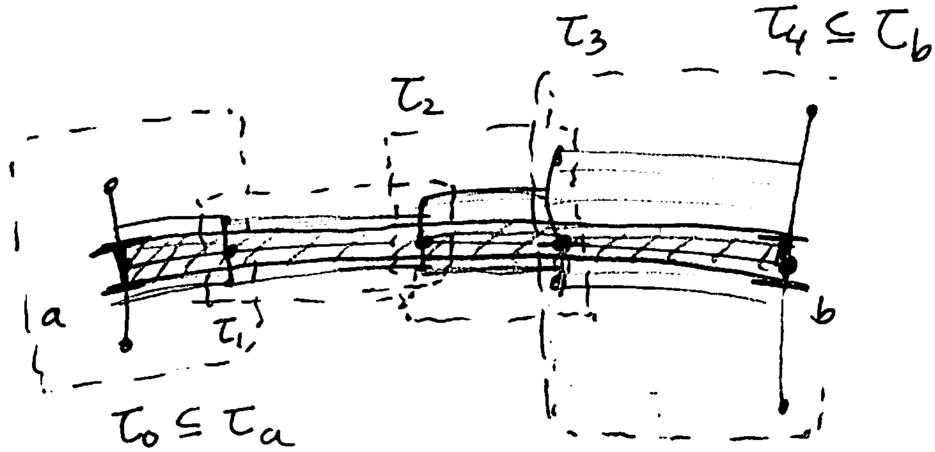
["sliding along the plaques", i.e. just use t coor]

Suppose α is a smooth path in a leaf L from a to b. Let τ_a, τ_b be trans. with a in the interior of τ_a, b in int of τ_b.

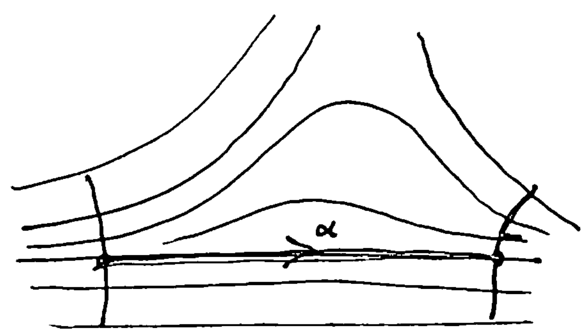
After shrinking τ_a, τ_b can define a diffeo $\mathrm{hol}_\alpha : \tau_a \longrightarrow \tau_b$ by "sliding along the leaves".

More precisely, take transversals τ_i so each meets α in its interior

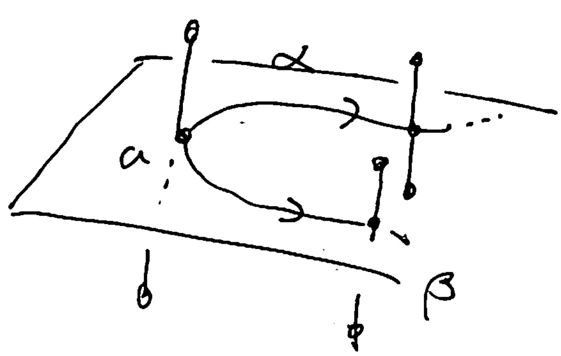
and each (τ_i, τ_{i+1}) is contained in a fol. chart.



Note: Restricting τ_a, τ_b is crucial

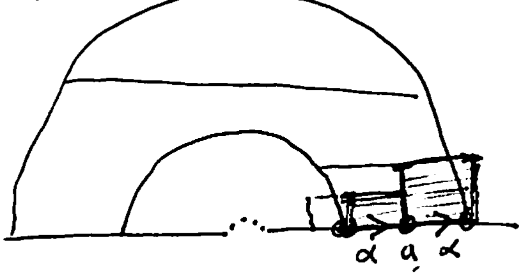


Lemma: Modulo shrinking the domain/ranges only depends on the relative homotopy class of α in L.

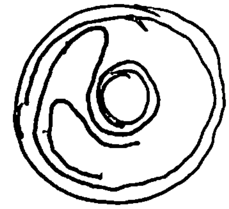


[Holonomy is a homomorphism from the groupoid of homotopy classes of paths in leaves to the groupoid of germs of of diffeos between transversals...]

Ex: Reeb annulus



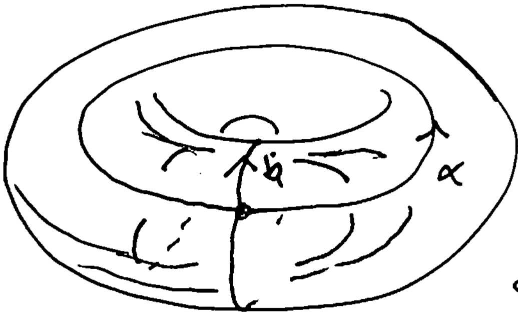
$\text{hol}_\alpha : t \mapsto \frac{1}{2} t$



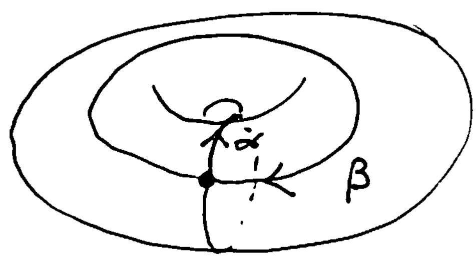
Ex: Reeb solid torus

$\text{hol}_\alpha : t \to \frac{1}{2} t$

$\text{hol}_\beta : t \mapsto t$



other solid torus



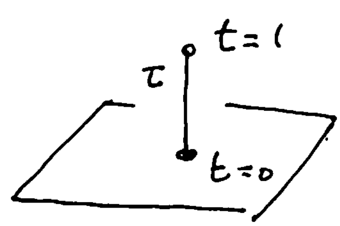
Back to S^3: Problem:

after gluing

$$\text{hol}_\alpha(t) = \begin{cases} \frac{1}{2} t & t \leq 0 \\ t & t \geq 0 \end{cases}$$

which is not smooth.

Def: Suppose $\mathcal{F}$ is a fol of M^3 with tangential boundary. Then $\mathcal{F}$ is infinitesimally trivial along a component L of ∂M when: For a transversal τ at $l_0 \in \partial M$ param by $[0, 1]$

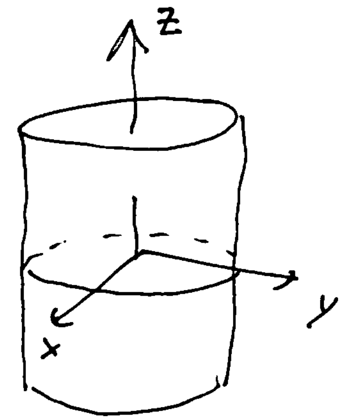


and for all $\alpha \in \pi_1(L, l_0)$ if $h = hol_\alpha$ is a diff $[0,a] \to [0,b]$ then $h'(0) = 1$ and $h^{(k)}(0) = 0$ for all $k > 1$.

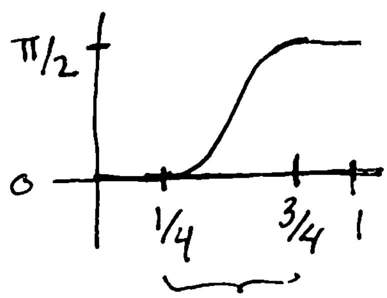
[That is, h looks like id at 0.]

<u>Non ex</u>: Our Reeb solid torus.

<u>Ex</u>: <u>Improved</u> Reeb solid torus.



Take a smooth λ on $[0,1]$.



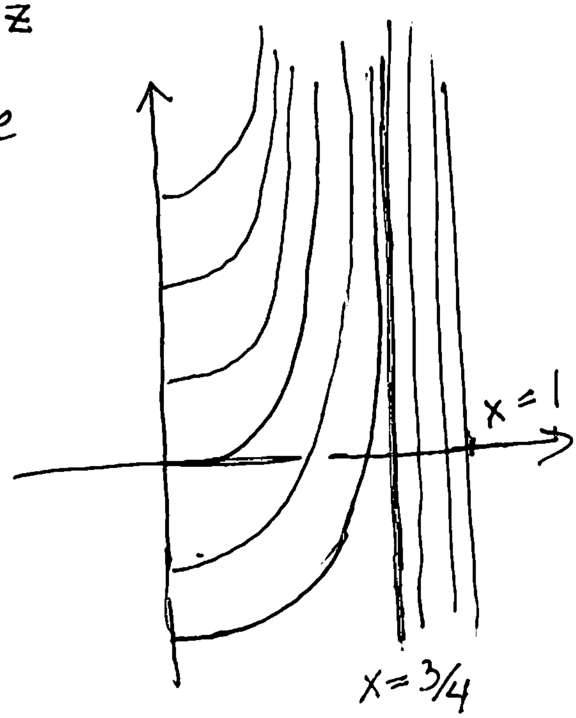
In cylindrical coor, set

$$\omega = -\sin(\lambda(r))\,dr + \cos(\lambda(r))\,dz$$

In (x,z) plane with $x \geq 0$ have

$$\cos(\lambda(r))\frac{\partial}{\partial x} + \sin(\lambda(r))\frac{\partial}{\partial z}$$

is a tangent to the leaves.



Set $M_0 = \{r \leq 1\} / (x,y,z) \to (x,y,z+1)$

and $M_1 \subseteq M_0$ where $\{r \leq 3/4\}$.

Then M_1 has a "Reeb fol" and

$\overline{M_0 \setminus M_1}$ is $[3/4,1] \times T^2$ fol by $\{pt\} \times T^2$.

Now $\mathcal{F}$ on M_1 is inf. trivial along ∂M_1 — just compute the derivatives of the holonomy focusing on the outside.

[Prop 3.4.2 of Fol I] Suppose N_i is foliated by $\mathcal{F}_i$ with S_i a comp of ∂N_i that is a leaf of $\mathcal{F}_i$. If both $\mathcal{F}_i$ are inf. trivial along S_i and $\varphi: S_1 \to S_2$ is a diffeo., then $N = N_1 \cup_\varphi N_2$ is smoothly fol by $\mathcal{F}_1 \cup \mathcal{F}_2$.

From now on, the improved Reeb comp will be the standard one. We can use it to foliate S^3 as desired.