Lecture 8: Cohomology of Product Spaces II.

Last time: Fix a ring R.

Künneth Thm: X, Y CW complexes with each $H^k(Y)$ a finitely gen free R-mod. Then

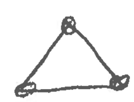
$$H^*(X) \otimes_R H^*(Y) \xrightarrow[\times]{} H^*(X \times Y)$$

is an isom. of rings.

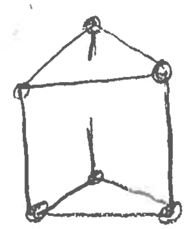
Today: Where does this come from?

Setting: X, Y finite CW complexes, $R = \mathbb{Z}$

$X \times Y$ has CW str with cells $a \times b$ with $a \subseteq X$, $b \subseteq Y$

$a =$ 

$b =$ 

$a \times b =$ 

$\partial(a \times b) = \partial a \times b \cup a \times \partial b$

For C, C' chain cplx's, define

$$C \otimes C' \text{ by } (C \otimes C')_n = \bigoplus_{k=0}^{n} C_k \otimes C'_{n-k}$$

$$\text{and } \bar{\partial}(c_k \otimes c'_\ell) = \partial c_k \otimes c'_\ell + (-1)^k c_k \otimes \partial' c'_\ell$$

[Can check $\bar{\partial}^2 = 0$, tensor of chain maps is a chain map, etc.]

<u>Lemma:</u> For cellular chains, have

$$C_*(X) \otimes C_*(Y) \xrightarrow{\cong} C_*(X \times Y)$$

$$a^k \otimes b^l \longmapsto a \times b$$

a^k, b^l cells; $a \times b$ $k+l$ cell.

<u>Unfun fact:</u> $H_*(C \otimes C')$ is not always $H_*(C) \otimes H_*(C')$!

$$C: \quad \overset{2}{\mathbb{Z}} \xrightarrow[\times m]{} \overset{1}{\mathbb{Z}} \to \overset{0}{0} \qquad H_1(C_*) = \mathbb{Z}/m \text{ others } 0$$

$$C': \quad 0 \longrightarrow \mathbb{Z} \xrightarrow[\times m]{} \mathbb{Z} \qquad H_0(C'_*) = \mathbb{Z}/m \text{ others } 0$$

But: $H_k(C_* \otimes C'_*) \cong \mathbb{Z}/m$ for $k = 1$ and 2.

<u>Thm.</u> C_*, C'_* chain comp. over a P.I.D. R with C_* free. Then for each n we have a nat'l exact seq

$$0 \longrightarrow \bigoplus_{k=0}^{n} H_k(C_*) \otimes_R H_{n-k}(C'_*) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow$$

$$\longrightarrow \bigoplus_{k=0}^{n-1} \mathrm{Tor}_R(H_k(C_*), H_{n-k-1}(C'_*)) \longrightarrow 0$$

which moreover splits.

can drop

<u>Cor:</u> Gen. Künneth thm for finite CW complexes

Cor: X, Y CW complexes, $R=$ field. Then each

$$H_n(X,Y) \cong \bigoplus_{k=0}^{n} H_k(X) \otimes_R H_{n-k}(Y).$$

Proof of Thm similar to that of U.C.T. See also Chap 7 of Munkres. There is also a gen. Künneth thm in cohom, final term is

$$\bigoplus_{k=0}^{n} \mathrm{Tor}_R \left(H^{k+1}(C_*), H^{n-k}(C'_*) \right)$$

Hatcher avoids most of this algebra by using axiom char. of cohomology. Fix Y and study

$$h^n(X) = \bigoplus_{k=0}^{n} H^k(X) \otimes_R H^{n-k}(Y)$$
$$k^n(X) = H^n(X \times Y)$$

[Assuming $H^*(Y)$ is a finitely gen and free in each dim.]

What about spaces which aren't CW complexes?

a) Use CW approx. $\underset{CW}{M} \xrightarrow{f} \underset{\text{some sp}}{X}$ f "almost" a hom. equiv.

[Eilenberg – Zilber] For any pair X, Y of top. spaces, there are chain maps

$$C_*^{\text{sing}}(X;\mathbb{Z}) \otimes C_*^{\text{sing}}(Y;\mathbb{Z}) \underset{\nu}{\overset{\mu}{\rightleftarrows}} C_*^{\text{sing}}(X \times Y;\mathbb{Z})$$

that are chain hom. inverses of one another; [they are nat'l w.r.t. to chain maps induc. by cont. maps.]

Pf uses the Acyclic Model Thm [§32 of Munkres]

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Division algebra $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow[\text{bilinear}]{} \mathbb{R}^n$ and $\forall a \neq 0, b \in \mathbb{R}^n$ both $ax = b$ and $xa = b$ are solvable ($\Leftrightarrow$ no zero divisors).

[Not assuming comm, assoc, unital, ...]

Ex: $\mathbb{R}, \mathbb{C}, \mathbb{H} = \mathbb{R}^4 = \langle 1, i, j, k \rangle$, $\mathbb{O} \cong \mathbb{R}^8$

$$i^2 = j^2 = k^2 = -1$$
$$ji = -ij$$

Thm If $\mathbb{R}^n$ has the structure of a div. alg, then $n = 2^k$. [In fact only $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.]

We'll need: [See text or wait a week or two.]

$$H^*(\mathbb{R}P^n; \underset{\mathbb{Z}/2\mathbb{Z}}{\mathbb{F}_2}) = \mathbb{F}_2[\alpha]/(\alpha^{n+1})$$

α is the gen of $H^1(\mathbb{R}P^n) \cong \mathbb{F}_2$

Note: $\alpha \cup \alpha = -\alpha \cup \alpha \Rightarrow 2(\alpha \cup \alpha) = 0$ is no info mod 2. In particular $H^*(X;\mathbb{F}_2)$ is a comm. ring.

Lemma: $H^*(\mathbb{R}P^n \times \mathbb{R}P^n;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha,\beta] / (\alpha^{n+1},\beta^{n+1})$

Idea: $X = \mathbb{R}P^n$, $Y = \mathbb{R}P^n$

α gen $H^1(X)$ $\quad$ β gen $H^1(Y)$

1_X gen $H^0(X)$ $\quad$ 1_Y gen $H^0(Y)$

$$H^1(X \times Y) \cong \underbrace{\left(H^1(X) \overset{\text{over } \mathbb{F}_2}{\otimes} H^0(Y)\right)}_{\cong \mathbb{F}_2 \text{ gen by } \alpha \otimes 1_Y} \oplus \underbrace{\left(H^0(X) \otimes H^1(Y)\right)}_{\cong \mathbb{F}_2 \text{ gen by } 1_X \otimes \beta}$$

$$= \mathbb{F}_2^2 \text{ gen by } \{\alpha \times 1_Y,\ 1_X \times \beta\}$$

$$H^2(X \times Y) = \langle \alpha^2 \times 1_Y,\ \alpha \times \beta,\ 1_X \times \beta^2 \rangle \cong \mathbb{F}_2^3$$

$\vdots$

etc.

[Proof of thm next time...]