Lecture 19: Other forms of duality

<u>Poincaré Duality</u>: M^n an R-orient mfld. Then each

$$D_M : H^k_c(M;R) \longrightarrow H_{n-k}(M;R) \text{ is an isom.}$$

Still need to show:

Ⓐ U, V open in M. If D_U, D_V, and $D_{U\cap V}$ are isom, so is $D_{U\cup V}$.

<u>Pf</u>: The following commutes up to sign:

$$\begin{tikzcd}
\to H^k_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"'] & H^k_c(U)\oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^k_c(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & H^{k-1}(U\cap V) \to \arrow[d, "D_{U\cap V}"] \\
\to H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U\cup V) \arrow[r] & H_{n-k-1}(U\cap V) \to
\end{tikzcd}$$

[Q: What's odd about the top row? Arrows are backwards!]

Top row follows from if $K \subseteq U$, $L \subseteq V$ are compact, then

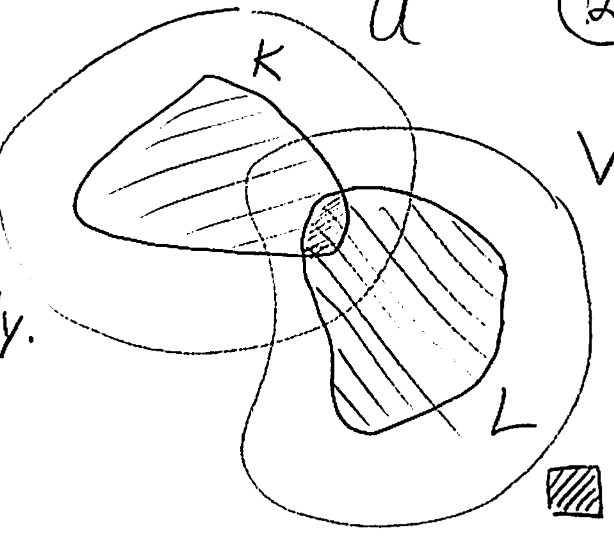
or M   or M   or M.

$$\to H^k(U\cap V \mid K\cap L) \to H^k(U \mid K) \oplus H^k(V \mid L) \to H^k(U\cup V \mid K\cup L) \to$$

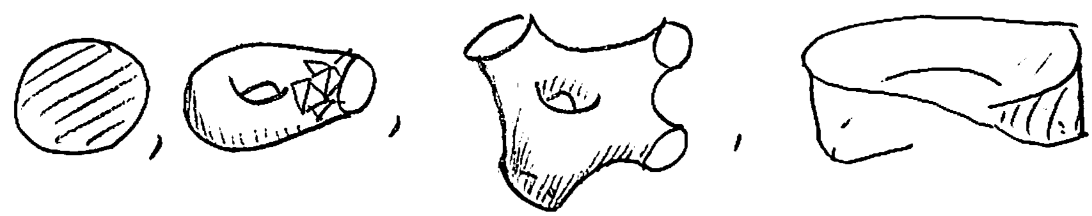
or M

is exact and moreover direct limits preserve exactness.

Bottom row is ordinary M-V, and vertical maps are direct limits. See Lemma 3.36 for proof of (almost) commutativity. Now get (A) by the 5-lemma.



A Hausdorff 2^{nd} count. topological space M is a <u>manifold with boundary</u> when every $x \in M$ has an open nbhd $\cong$ to $\mathbb{R}^n$ or $\mathbb{R}^{n-1} \times [0, \infty)$

<u>n = 2</u>: 

If $\partial M \neq \phi$, then $H_n(M) = 0$ since $M \simeq_{h.e.} M \setminus \partial M$ and non-cpt mflds have $H_n(M) \cong H^0_c(M) = 0$.

<u>Thm</u>: M^n a cpt R-orient mfld w/ bdry. Then

(A) $H^k(M) \cong H_{n-k}(M, \partial M)$

(B) $H^k(M, \partial M) \cong H_{n-k}(M)$.

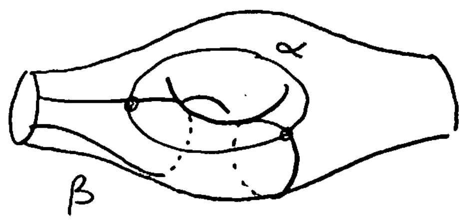
<u>Cor</u>: If M is connected, then $H_n(M, \partial M) \cong R$ and get a rel. fund class $[M]$. Isoms given by $\varphi \mapsto [M] \cap \varphi$.

Geometric picture: [cap prod on homology.]

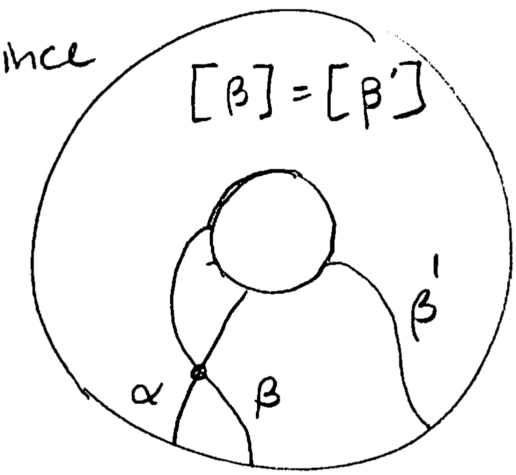
$$H_1(S) \times H_1(S,\partial S) \longrightarrow \mathbb{Z}$$
$\alpha \qquad \beta \qquad \alpha \cap \beta$

works but not

$$H_1(S,\partial S) \times H_1(S,\partial S) \to \mathbb{Z}$$



since

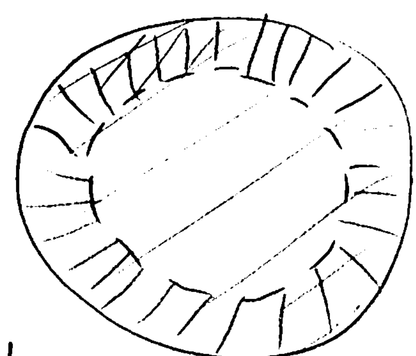


So $H^1(S) = \mathrm{Hom}(H_1(S), \mathbb{Z})$
$\qquad \cong H_1(S, \partial S)$.

[Can prove either way, will use 2nd proof.]

Prop: If M is a cpt mfld with bdry, then ∂M has an open nbhd $\cong$ to $M \times [0,1)$

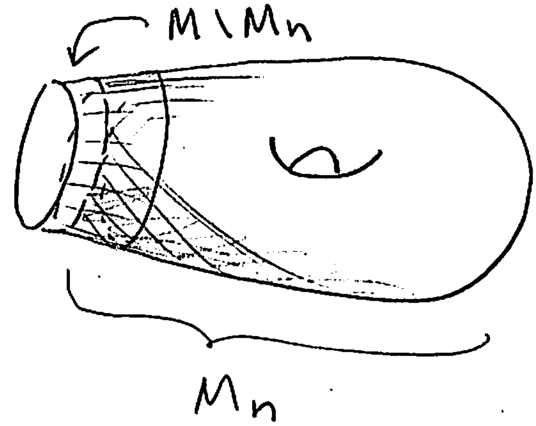
[Pf: see Hatcher.]



Cor: $N = M \setminus (\partial M \times [0, 1/2))$ is a def retract of M and $M \setminus \partial M$. In particular, $M \setminus \partial M \simeq_{h.e} M \cong N$.

Pf of thm: (B) Follows from $H^k(M, \partial M) \cong H^k_c(M \setminus \partial M)$ by the Cor and usual P.D. In more detail, set

Set $M_n = M \setminus (\partial M \times [0, 1/n])$



If $\mathring{M} = M \setminus \partial M$, then

$$H^k_c(\mathring{M}) = \varinjlim_n H^k(\mathring{M} \mid M_n)$$

and $H^k(\mathring{M} \mid M_n) \cong H^k(M \mid M_n) = H^k(M, M \setminus M_n)$
$\cong H^k(M_n, \partial M_n) \cong H^k(M, \partial M)$

so $H^k_c(\mathring{M}) \cong H^k(M, \partial M)$. Also, $H_{n-k}(M) \cong H_{n-k}(\mathring{M})$ by Cor.

Ⓐ follows from Ⓑ via long exact sequences of the pair, which are compat. with $[M] \cap \cdot$ and use

$H_n(M, \partial M) \to H_{n-1}(\partial M)$ sends $[M]$ to $[\partial M]$. ▨

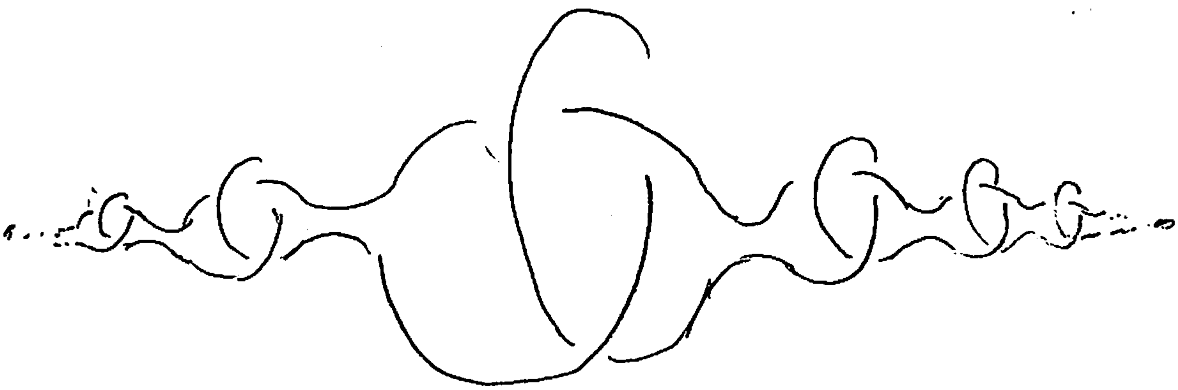
See text for details.

<u>Alexander Duality</u>: If $K \subseteq S^n$ is cpt, locally contractible, and not $\emptyset$ or S^n, then for all i: $\tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$

<u>Cor</u>: $\tilde{H}_*(S^n \setminus K)$ does <u>not</u> depend on how K is embedded in S^n.

$S^1 \hookrightarrow S^3$:  or

$\pi_1(S^3 \setminus K)$ is not finitely gen:



In each case, $\tilde{H}^k(S^3 \setminus S^1) \cong \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \text{otherwise} \end{cases}$