More on fiber bundles.

Suppose G is a topological group acting on a space X.

A **fiber bundle with fiber F and structure group G** is a map $p: E \to B$ together with a collection of homeom.

$\{\varphi : p^{-1}(U) \to U \times F\}$ for certain $U \subseteq B$ where:

1) The U cover B.

2) Each
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{proj}_U"] \\ & U & \end{tikzcd}$$
commutes

3) If (U, φ) and (V, ψ) are charts, $\exists$ a continuous map $\theta : U \cap V \to G$ so that $\forall\, x \in U \cap V$ and $f \in F$ have
$$(x, \theta(x) \cdot f) = \varphi(\psi^{-1}((x, f)))$$

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4) If (U, φ) is a chart and $V \underset{\text{open}}{\subseteq} U$ so is $(V, \varphi|_V)$.

5) The collection (U, φ) is maximal with respect to (1-4).

6) G acts effectively on F, i.e. $G \hookrightarrow \mathrm{Homeo}(F)$.

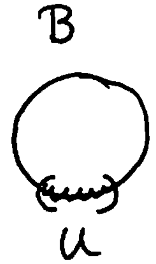
Principal bundles: G = any top gp, $F = G$ acted on by left trans.

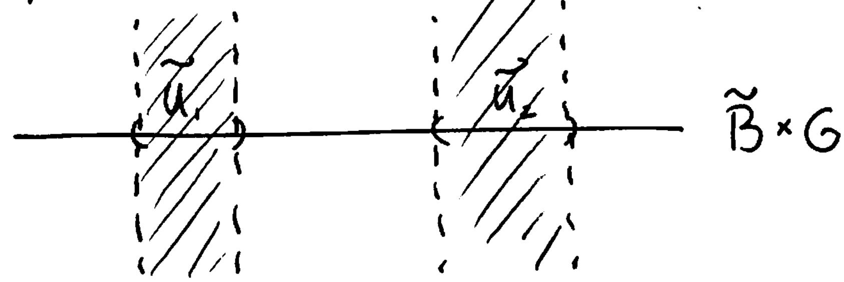
Construction: $\rho : \pi_1 B \to G$ $\quad \tilde{E} = \tilde{B} \times G$ acted on by $\pi_1 B$ via $\gamma \cdot (\tilde{b}, g) = (\gamma \cdot \tilde{b}, \rho(\gamma) \cdot g)$

$$E = \tilde{B} \times G \,/\, \pi_1 B$$

$$p : E \to B$$
$$[(\tilde{b}, g)] \mapsto \pi(\tilde{b})$$

This gives a <u>flat</u> principle bundle, since θ is locally constant.





$$U\times G \xleftarrow[\substack{\varphi_1 \\ \text{cor to } \tilde{U}_1}]{} p^{-1}(U) \xrightarrow[\substack{\varphi_2 \\ \text{cor to } \tilde{U}_2}]{} U\times G$$

Then $\theta : U \to G$ is the const $\rho(\gamma)$, where $\gamma \in \pi_1 B$ sends $\tilde{U}_2 \to \tilde{U}_1$. Then $\varphi_1 \circ \varphi_2^{-1}(x, g) = (x, \theta(x)\cdot g)$.

<u>Note:</u> For a flat princ. bundle, if U is simply conn then an ident of $p^{-1}(x_0 \in U)$ with G gives a unique trivialization of $p^{-1}(U)$ as $U\times G$.

<u>Cf</u>: Flat connection.

<u>Generalization</u>: G acts on F, $\rho : \pi_1 B \to G$.

Get a fiber bundle $E = \pi_1 B \backslash \tilde{B}\times F \to B$. where $\gamma \in \pi_1 B$

acts by $(\tilde{b}, f) \mapsto (\gamma\cdot\tilde{b}, \rho(\gamma)\cdot f)$

<u>Ex</u>: $B = S^1$, $G = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$, $F = [-1, 1]$ $\quad g\cdot f = gf$

Then $E(\rho, F) = ??$

If G is discrete, a principal G-bundle is a regular covering space corresponding to a homomorphism $\pi_1 B \to G$.

Pull backs: $p: E \to B$ fiber bundle. Given $f: A \to B$, define $f^*(E) = \underset{\text{in } A \times E}{\{(a,e)} \mid f(a) = p(e)\}$. HW: The map $\pi_A : f^*(E) \to A$ is a fiber bundle with the same fiber and structure gp as $E \to B$.

Universal Bundles: If G is a topological group, there exists a principal G-bundle $EG \to BG$ so that $\forall$ CW's X we have

$$[X, BG] \cong \begin{array}{c}\text{Isomorphism classes} \\ \text{of principal } G\text{-bundles} \\ \text{over } B\end{array}$$

$$(f: X \to BG) \longmapsto f^*(EG)$$

If G has the discrete topology, then $BG = K(G,1)$ and $EG \to BG$ is as in the 1st proof of the existence of $K(G,1)$'s.

Ex: $G = \mathbb{Z}$ with the discrete topology.

$BG = K(\mathbb{Z}, 1) = S^1$

Q: What is EG? A: $\mathbb{R}$

For any X, what map $f: X \to S^1$ gives the trivial bundle? It's the const map.

Ex: $G = S^1$ with usual top. Then $BG = \mathbb{C}P^\infty$.

Recall
$$\begin{array}{ccc} S^1 \hookrightarrow & S^{2n+1} & \\ & \downarrow & \\ & \mathbb{C}P^n & \end{array}$$
by viewing $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ and S^1 acts diagonally.

Then have
$$\begin{array}{ccc} S^1 \hookrightarrow & S^\infty & \\ & \downarrow & \\ & \mathbb{C}P^\infty & \end{array}$$
as the univ. S^1-bundle ($= EG$).

Note: Cohomology of BG leads to <u>characteristic classes</u> of fiber bundles. E.g. any S^1-fiber bundle has an Euler class $\in H^2(X; \mathbb{Z})$ comming from the gen of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

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Next topic: Homology with local coeffs (§3.H in Hatcher)

Bundle of gps: $p: E \to B$ with fiber a discrete abelian gp G and str. gp $\mathrm{Aut}(G)$ with the discrete top.

Ex: M an n-mfld. R a ring.
$$\begin{array}{ccc} M_R & = & \{\alpha_x \in H^n(M, M \setminus \{x\}; R) \mid x \in M\} \\ \downarrow & & \downarrow \\ M & & x \end{array}$$