Math 526: HW 7 due Friday, December 10, 2021.

1. Suppose $p\colon E \to B$ is a fiber bundle with fiber F and structure group G. Given $f\colon A \to B$ define $f^*(E) = \{(a,e) \in A \times E \mid f(a) = p(e)\}$. Prove that the projection $\pi_A\colon f^*(E) \to A$ is a fiber bundle with same fiber and structure group as $E \to B$; it is called the *pullback bundle* of $E \to B$ under f.

2. Let US^2 be the unit tangent bundle to S^2. Concretely,

$$US^2 = \left\{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; |x| = |v| = 1 \text{ and } x \cdot v = 0\right\}$$

More abstractly, US^2 is the subspace of the tangent bundle TS^2 consisting of tangent vectors that are unit-length with respect to the usual round Riemannian metric.

   (a) Show that the map $p\colon US^2 \to S^2$ is a principal bundle with structure group $S^1 = \mathrm{SO}(2)$.

   (b) Prove that US^2, $\mathrm{SO}(3)$, and $\mathbb{RP}^3$ are all homeomorphic.

3. Regard $\mathbb{CP}^n$ as the set of lines in $\mathbb{C}^{n+1}$. Consider $E^n = \{(z,L) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n \mid z \in L\}$ which has a natural projection $p_n\colon E^n \to \mathbb{CP}^n$. Prove that $p_n\colon E^n \to \mathbb{CP}^n$ is a complex vector bundle where the fibers are copies of $\mathbb{C}$. Such bundles are called “complex line bundles”.

   Contextual note: Taking unions, we get a complex line bundle $p\colon E \to \mathbb{CP}^\infty$. It turns out that this is the universal line bundle in the following sense. By problem (a), given $f\colon B \to \mathbb{CP}^\infty$ we get a complex line bundle $f^*(E)$ over B. In fact, isomorphism classes of complex line bundles over B are in bijective correspondence with $[B, \mathbb{CP}^\infty] \cong H^2(B;\mathbb{Z})$. The cohomology class associated to a complex line bundle over B is called the first Chern class.

4. Fix once and for all three distinct points a_1, a_2, a_3 in S^2. Let $B = S^2 \setminus \{a_1, a_2, a_3\}$. Given $b \in B$, let T_b be the 2-fold cover of $B \setminus \{b\}$ so that a small loop about one of the deleted points $\{a_1, a_2, a_3, b\}$ does *not* lift to T_b.

   (a) Show that T_b is a 2-torus $\overline{T}_b$ with four points deleted. Show also that the covering map $T_b \to B \setminus b$ extends to a continuous map of $\overline{T}_b \to S^2$ which looks like $z \mapsto z^2$ on $\mathbb{C}$ near each of $\{a_1, a_2, a_3, b\}$.

   Note: The map $\overline{T}_b \to S^2$ is an example of a branched or ramified covering map. This particular map is often called the quotient of the 2-torus by the elliptic involution.

   (b) Show that you can build a bundle $p\colon E \to B$ with fiber $\mathbb{Z}^2$ and structure group $\mathrm{GL}_2\mathbb{Z}$ by taking $p^{-1}(b) = H_1(\overline{T}_b;\mathbb{Z})$.

   (c) Prove that E is not isomorphic to the trivial bundle $B \times \mathbb{Z}^2$.

   (d) Prove or disprove: $H_*(B;E) = 0$.