Lecture 22: Properties of the determinant.

[§ 4.2 of FIS]

Last time: For $A \in M_{n\times n}(\mathbb{R})$ we defined

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det(\tilde{A}_{1j})$$

$(n-1)\times(n-1)$ matrix gotten by deleting row 1 and column j from A.

Thm: det is a linear function of the r^{th} row when all the other rows are fixed.

Thm: For $A \in M_{n\times n}(\mathbb{R})$ and $1 \le r \le n$ one has

$$\det(A) = \sum_{j=1}^{n} (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj})$$

Explained last time how the cofactor expansion thm follows from

Lemma: Let $B \in M_{n\times n}(\mathbb{R})$ for $n \ge 1$. If row r of B is e_j, then $\det(B) = (-1)^{r+j} \det(\tilde{B}_{rj})$.

Convention: $\det(0\times 0 \text{ matrix}) = 1$.

Proof of Lemma: See page 214 of [FIS].

Similar to proof of first thm in that we induct on n. Gets a little messy...

[Discuss alternate viewpoints.]

$$\begin{array}{cc} & \begin{pmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1n} \\ & & \vdots & & \\ 0 & \cdots 0 & & 0 \cdots & 0 \\ & & \vdots & & \end{pmatrix} \\ r & \\ & \qquad\qquad\quad j \end{array}$$

Cor: If A has a row of all zeros, then $\det(A) = 0$.

Pf: Suppose row r of A is 0. As det is a linear fn of row r when all other rows are fixed, and any linear fn send 0 to 0, we must have $\det(A) = 0$. $\square$

Cor: If $A \in M_{n \times n}(\mathbb{R})$ has two identical rows, then $\det(A) = 0$.

Proof: Clear for $n = 2$, so let's induct on n. If $s_1 \neq s_2$ are the indices of two matching rows, expand along a row r where $r \neq s_i$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj})$$

also has two identical rows

$$= 0$$

since all $\det(\tilde{A}_{rj}) = 0$ by induction. ∎

[Key to understanding det: row ops.]

How does det change under row ops?

① $A \xrightarrow{R_r \leftrightarrow R_s} B \implies \det(B) = -\det(A)$

Pf: Can assume $s = 1$ as any such row op can be implemented by three type ① row ops involving the first row:

$$\begin{tikzcd} A \arrow[r, "R_r \leftrightarrow R_1"] \arrow[rrr, bend right, leftrightarrow, "R_r \leftrightarrow R_s"'] & A' \arrow[r, "R_s \leftrightarrow R_1"] & A'' \arrow[r, "R_r \leftrightarrow R_1"] & B \end{tikzcd}$$

Suppose $A = \begin{pmatrix} - & a_1 & - \\ & \vdots & \\ - & a_r & - \\ & \vdots & \end{pmatrix}$ so $B = \begin{pmatrix} - & a_r & - \\ & \vdots & \\ - & a_1 & - \\ & \vdots & \end{pmatrix}$

By corollary, have

$$0 = \det\begin{pmatrix} — a_1 + a_r — \\ — a_2 — \\ \vdots \\ — a_1 + a_r — \\ \vdots \end{pmatrix} = \det\begin{pmatrix} — a_1 — \\ — a_2 — \\ \vdots \\ — a_1 + a_r — \end{pmatrix} + \det\begin{pmatrix} — a_r — \\ — a_2 — \\ \vdots \\ — a_1 + a_r — \end{pmatrix}$$

$$= 0 + \underbrace{\det\begin{pmatrix} — a_1 — \\ \vdots \\ — a_r — \\ \vdots \end{pmatrix}}_{A} + \underbrace{\det\begin{pmatrix} — a_r — \\ — a_2 — \\ \vdots \\ — a_1 — \\ \vdots \end{pmatrix}}_{B} + 0$$

∎

② $A \xrightarrow{cR_r} B \implies \det(B) = c\det(A)$

Pf: Row-wise linearity.

③ $A \xrightarrow{kR_s + R_r} B \implies \det(A) = \det(B)$

Pf: Suppose C is equal to A except row r is replaced by a copy of row s. Then A, B, C are the same except for row r where

$$(\text{row } r \text{ of } B) = (\text{row } r \text{ of } A) + k(\text{row } r \text{ of } C)$$

As $\det(C) = 0$ as it has a repeated row, we get $\det(B) = \det(A) + k\det(C) = \det(A)$.

∎

Can use to compute the determinant.

<u>Ex</u>:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{-R_1 + R_2}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{-2R_1 + R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{\frac{1}{2} R_2}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1/2 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{-R_2 + R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -7/2 \end{pmatrix} = C$$

<u>Thm</u>: <u>If</u> A is upper or lower diagonal, then $\det(A)$ = product of the diagonal entries.

So $\det(C) = -7/2$ and $\det(C) = -\frac{1}{2}\det(A)$

$\Rightarrow \det(A) = 7$ as we got last time.

For larger matrices, this is the only way to go: cofactor expansion takes $> n!$ arithmetical operations whereas row reduction takes $C \cdot n^3$.

Even Blue Waters would take $> 10^{100}$ years to compute a 100×100 det by the first method; my laptop took 10^{-2} seconds!

<u>Proof of Thm:</u> If $A = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & 0 & \cdots & & \\ A_{31} & A_{32} & A_{33} & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \\ A_{n1} & \cdots & & & & A_{nn} \end{pmatrix}$

then

$$\det(A) = A_{11} \det(\underbrace{\tilde{A}_{11}}_{\text{Also lower triangular}})$$

$$= \ldots \text{induct on } n \ldots = A_{11} A_{22} \cdots A_{nn}.$$

Next time will show $\det(A^t) = \det(A)$ which gives the upper triangular case.