Lecture 26: Finding eigenvectors [§5.1 and §5.2]

Last time: An eigenvector for a linear operator T on V is a $v \in V$ where $T(v) = \lambda v$ for some scalar λ.
eigenvalue assoc to v.

An eigenvector for $A \in M_{n\times n}(\mathbb{R})$ is a $v \in \mathbb{R}^n$ with $Av = \lambda v$. [Equivalently, its an eigenvector for L_A.]

[Today will explain how to find all eigenvectors for A. Recall:]

Thm: Suppose $A \in M_{n\times n}(\mathbb{R})$. Then $\lambda \in \mathbb{R}$ is an eigenvalue for A if and only if $\det(A - \lambda I_n) = 0$.

Def: The characteristic polynomial of $A \in M_{n\times n}(\mathbb{R})$ is $f(t) = \det(A - tI_n)$.

Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$ $\quad A - \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} -t & 0 & 1 \\ 0 & 2-t & 0 \\ -2 & 0 & 3-t \end{pmatrix}$

$$\det(A - t) = (2-t)(-t(3-t) + 2)$$
$$= (2-t)(t^2 - 3t + 2) = -t^3 + 5t^2 - 8t + 4$$

Notes: Suppose $f(t)$ is the characteristic poly of $A \in M_{n \times n}(\mathbb{R})$.

1) For any $\lambda \in \mathbb{R}$, have $f(\lambda) = \det(A - \lambda I_n)$.

Thus

Thm: The roots of $f(t)$ are exactly the eigenvalues of A.

Ex: A as before, have char poly

$$\begin{aligned} f(t) &= (2-t)(t^2-3t+2) \\ &= -(t-2)\big((t-2)(t-1)\big) \\ &= -(t-1)(t-2)^2 \end{aligned}$$

So the eigenvalues of A are 1 and 2.

2) $f(t)$ has degree n and leading coefficient $(-1)^n$.

Cor: An $n \times n$ matrix has at most n distinct eigenvalues

<u>Lemma</u> (Last time) $v \in \mathbb{R}^n$ is an eigenvector for $A \in M_{n\times n}(\mathbb{R}) \iff v \in \mathcal{N}(A - \lambda I_n)$ and $v \neq 0$.

<u>Ex</u>: Find all eigenvectors of $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$ with eigenvalue 1. Set $B = A - 1 \cdot I_3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix}$

As $B \xrightarrow{-2R_1 + R_3} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, have one free var and

$$\mathcal{N}(B) = \{(t, 0, t) \mid t \in \mathbb{R}\}$$
$$= \operatorname{span}\{(1, 0, 1)\}$$

So the eigenvectors of A with eigenvalue 1 are $(t, 0, t)$ with $t \neq 0$.

<u>Check</u>: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ✓

<u>Def</u>: If λ is an eigenvalue of $A \in M_{n\times n}(\mathbb{R})$, define $E_\lambda = \mathcal{N}(A - \lambda I_n)$, which is called the <u>eigenspace of A corresponding to λ</u>.

Note: $E_\lambda = \left\{ \begin{array}{c} \text{all eigenvectors of } A \\ \text{with eigenvalue } \lambda \end{array} \right\} \cup \{0\}$

and E_λ is a subspace of $\mathbb{R}^n$.

Ex: Compute E_2 for $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$.

$$\text{Set } B = A - 2I_3 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$E_2 = \mathcal{N}(B) = \{ (t/2, s, t) \mid s, t \in \mathbb{R} \}$$

Q: Is A diagonalizable? That is, can we find a basis β of $\mathbb{R}^3$ so that $[L_A]_\beta$ is a diagonal matrix?

A. Yes is equivalent to having a basis of $\mathbb{R}^3$ consisting of eigenvectors.

$$v_1 = (1, 0, 1) \in E_1$$

$$\left.\begin{array}{l} v_2 = (0, 1, 0) \\ \\ v_3 = (1, 0, 2) \end{array}\right\} \in E_2$$

Then $\beta = \{v_1, v_2, v_3\}$ are a basis as if

$$Q = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{then } \det(Q) = 1 \neq 0.$$

So

$$[L_A]_\beta = [I_{\mathbb{R}^3}]_{std}^{\beta} [L_A]_{std} [I_{\mathbb{R}^3}]_{\beta}^{std}$$
$$= Q^{-1} A Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and hence A is diagonalizable.

Note: $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if A is similar to a diagonal matrix.

Today focused on matrices, but can also consider

Def: Suppose T is a linear operator on V.
If λ is an eigenvalue of T, then set
$E_\lambda = \{v \in V \mid T(v) = \lambda v\}$, which

is called the eigenspace of T corresponding to λ. Note E_λ is a subspace of V and consists of all eigenvectors of T with this eigenvalue together with 0.

Exercise: $E_\lambda(A) = E_\lambda(L_A)$.

Application: For A as above and $n \geq 1$ we have $A^n = \begin{pmatrix} -2^n+2 & 0 & 2^n-1 \\ 0 & 2^n & 0 \\ -2^{n+1}+2 & 0 & 2^{n+1}-1 \end{pmatrix}$

so for example $A^8 = \begin{pmatrix} -254 & 0 & 255 \\ 0 & 256 & 0 \\ -510 & 0 & 511 \end{pmatrix}$

Reason: Set $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ so that $Q^{-1}AQ = D$ and so $A = QDQ^{-1}$. Then

$$A^n = A\,A\,A \cdots A = (QD\underbrace{Q^{-1})(Q}_{=I_n}DQ^{-1}) \cdots (QDQ^{-1})$$

$$= Q\,D\,D \cdots D\,Q^{-1} = Q D^n Q^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$ which expands

to the claimed expression for A^n.