Lecture 2: Vector spaces

[Go over syllabus.]

Ex: Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$, or indeed $\mathbb{R}^n$.

Def: A vector space over $\mathbb{R}$ is a set V with two operations

Addition: Assigns to each pair v, w in V a unique $v+w$ in V.

Scalar mult: Assigns to each a in $\mathbb{R}$ and v in V a unique av in V.

where the following rules hold.

1) For all u, v in V, $u+v = v+u$

2) For all u, v, w in V, $(u+v)+w = u+(v+w)$

3) There is an elt of V, called "0", so that for all v in V, $v+0 = v$.

4) For all v in V there exist w in V with

$$v + w = 0$$

5) For all v in V, $1v = v$.

6) For all a, b in $\mathbb{R}$ and v in V, $(ab)v = a(bv)$

7) For all a in $\mathbb{R}$ and u, v in V:

$$a(u+v) = au + bv$$

8) For all a, b in $\mathbb{R}$ and v in V, $(a+b)v = av + bv$

<u>Example</u>: $\mathbb{R}^n$ with coordinate-wise addition and scalar mult.

[Check one rule, choosen by the class.]

<u>Example</u>:

$$\mathrm{Mat}_{m\times n} = \left\{ \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}}_{}{}^{m\times n \text{ matrix}} \text{ where } a_{ij} \text{ are in } \mathbb{R} \right\}$$

where addition and scalar mult are again componentwise.

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 1 \end{pmatrix} + \underbrace{2\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \end{pmatrix}}_{\begin{pmatrix} 2 & 2 & 4 \\ 0 & -2 & 6 \end{pmatrix}} = \begin{pmatrix} 3 & 2 & 7 \\ 0 & 3 & 7 \end{pmatrix}$$

<u>Example</u>: $\mathcal{F} = \{\text{Continous fns from } [-1,1] \text{ to } \mathbb{R}\}$

$f+g$ is the fn where $(f+g)(x) = f(x) + g(x)$.

af is the fn where $(af)(x) = a f(x)$

[Some aspects of vectors in 2 and 3d are not part of this definition (no dot product, for ex), however, many familiar properties do follow from these rules.] For example,

<u>Question</u>: Is $\underset{\text{in } \mathbb{R}}{0} \cdot v = \underset{\text{in } V.}{0}$?

<u>Thm</u>: If u, v, w are in a vector space V and $u + w = v + w$, then $u = v$.

<u>Proof</u>: By ④, there is a z in V with $w + z = 0$. So

$$u \underset{③}{=} u + 0 = u + (w + z) \underset{②}{=} (u + w) + z$$

$$\underset{\text{Hypothesis}}{=} (v + w) + z \underset{②}{=} v + (w + z) = v + 0 \underset{③}{=} v$$

Thm: If v is in a vector space V, then $0v = 0$ in V.

Proof: We have

$$0v + 0v \underset{8}{=} (0+0)v = 0v \underset{3}{=} 0v + 0$$

$$\underset{1}{=} 0 + 0v.$$

By the previous theorem, this gives

$0v = 0$.

Related facts (see text and HW)

a) The 0 vector is unique.

b) The vector w with $v + w = 0$ is unique; we'll call it "$-v$". Note

$-v = (-1)v$ as

$$v + (-1)v \underset{5}{=} 1v + (-1)v \underset{8}{=} (1-1)v = 0 \cdot v \overset{\text{By above.}}{=} 0.$$

Sometimes, will allow scalars other than $\mathbb{R}$, most commonly the complex numbers $\mathbb{C} = \{a+bi\}$ where a, b are in $\mathbb{R}$ and $i^2 = -1$.

Ex: $V = \mathbb{C}^2 = \{(z_1, z_2) \text{ where } z_i \text{ in } \mathbb{C}\}$

$$(2+i, 3) + \underbrace{(1+i)(1-i, 3i)}_{(2,\, 3i-3)} = (4+i, 3i)$$

[Useful for math math and physical appications.]

More generally, can define a vector space over any field $\mathbb{F}$, which is a set with operations $(+, \times, -, \div)$ satisfying a bunch of axioms.

Ex: Field of two elts $\{0, 1\}$ where

$$\begin{aligned} 0+0 &= 0 \\ 0+1 &= 1 \\ 1+0 &= 1 \\ 1+1 &= 0 \end{aligned}$$

and

$$\begin{aligned} 0 \times 0 &= 0 \\ 0 \times 1 &= 0 \\ 1 \times 0 &= 0 \\ 1 \times 1 &= 1 \end{aligned}$$

[Here, $-$ is the same as $+$ and $\div$ is the same as $\times$
Finite fields are important in cryptography and coding theory, and are featured in Math 417.]

For the first part of the course we will always use $\mathbb{R}$ for the scalars, but [FIS] uses the language of fields.

See Appendix C of [FIS] for more on fields.