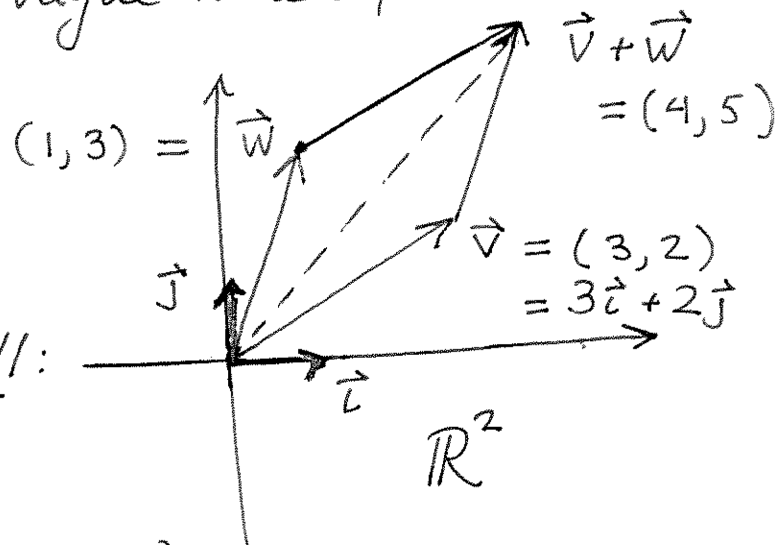
Math 416: Abstract Linear Algebra

Lecture 1: Introduction

[Fill out surveys
Go over syllabus at end of class]

Main topics: Vector spaces and linear transformations.

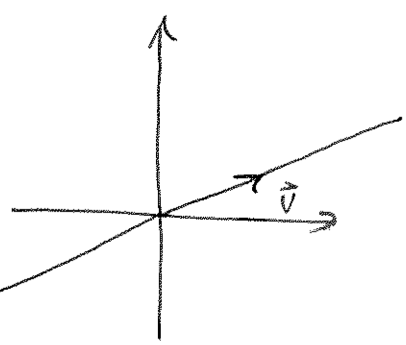
[and friends like matrices, linear equations...]
Today's goal is to give a vague idea of what these are.



Vectors in 2d:

Previously on Math 241:
[Query!]

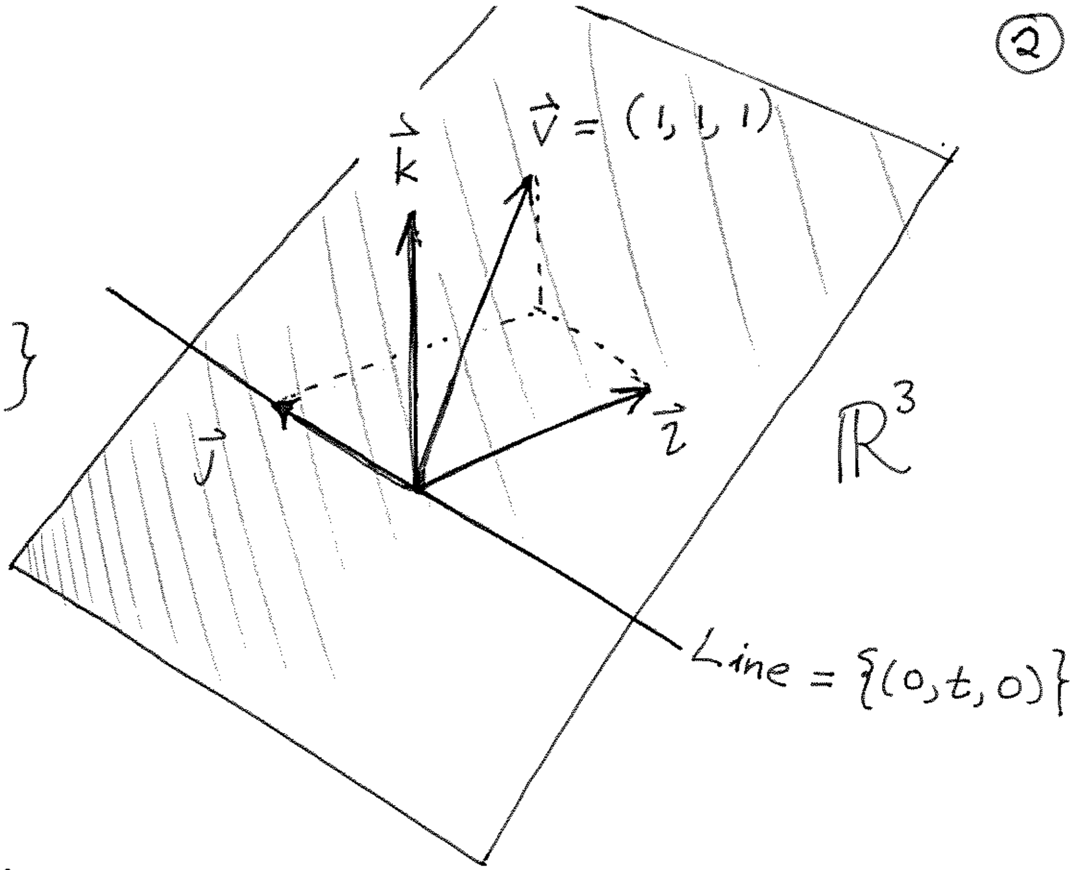
- scalar mult $2 \cdot \vec{v} = (6,4)$
- dot product $\vec{v} \cdot \vec{w} = 9 = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$
- area(w v) $= \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix} = 7$
- lines



$L = \{ t\vec{v} \text{ for } t \text{ a real } \# \}$

Vectors in 3^d:

Plane $\{x - z = 0\}$ containing $\vec{v}, \vec{J}$



Vectors in n-dimensions:

$\mathbb{R}^n = n$ tuples of real #s $= \{(a_1, \ldots, a_n)$ with a_i in $\mathbb{R}\}$

Add tuples componentwise

$$(1,3,0,-2) + (1,0,-1,3) = (2,3,-1,1)$$

and scale tuples like

$$3 \cdot (1,3,0,-2) = (3,9,0,-6)$$

Vector Space: Set V where we can add any pair of elts v_1 and v_2 to get another element of V, plus a notion of scalar mult. [Satisfies various rules...]

Why study?

1. Tuples in $\mathbb{R}^n$ occur naturally wherever you collect data.

Record high temp at 6 cities each day

$$\vec{V}_{\text{Jan }19} = (\underset{\text{Urbana}}{18}, \overset{\text{Boston}}{25}, \underset{\text{LA}}{64}, \overset{\text{Rio}}{75}, \underset{\text{Shenzhen}}{66}, \overset{\text{Murmansk}}{-11})$$

for one year $\rightsquigarrow$ 365 points in $\mathbb{R}^6$.

If we group by city

$$\vec{W}_{\text{Urbana}} = (32, 36, 33, 35, 35, 41, \ldots, \overset{19\text{th entry}}{18}, \ldots)$$

get 6 vectors in $\mathbb{R}^{365}$.

This is peanuts in the age of Big Data

[Medium data: word2vec: 400k pts in $\mathbb{R}^{300}$]

2. Real world multivariable functions.

price of oats (rainfall, start of growing season, end of growing season, pop of breakfast cereal, ...)

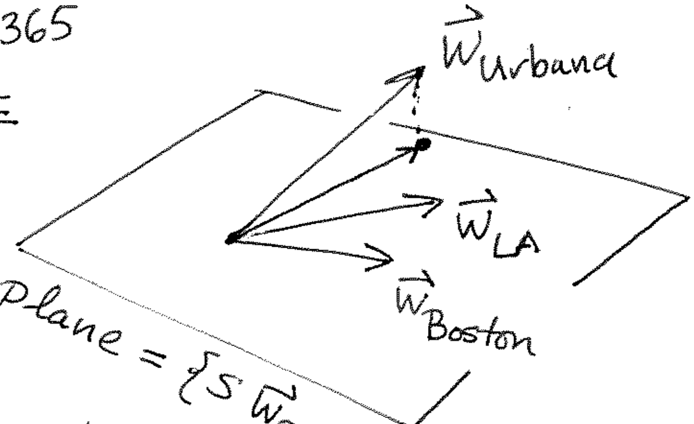
3 **Power of abstraction:** Use our intuition about 2 and 3 dimensions to understand things we can't possibly visualize.

a **Linear regression:** Find c_1 and c_2 so that

$$\begin{pmatrix}\text{High in} \\ \text{Urbana}\end{pmatrix} \approx c_1 \begin{pmatrix}\text{High in} \\ \text{Boston}\end{pmatrix} + c_2 \begin{pmatrix}\text{High in} \\ \text{LA}\end{pmatrix}$$

over the whole year.

$\underline{\underline{\mathbb{R}}}^{365}$



$\text{Plane} = \{s\,\vec{w}_{Boston} + t\,\vec{w}_{LA}\}$

If $\vec{w}_{Urbana}$ in this plane, get exact match. Otherwise find closest point (projection) and use that to get "best fit" (c_1, c_2).

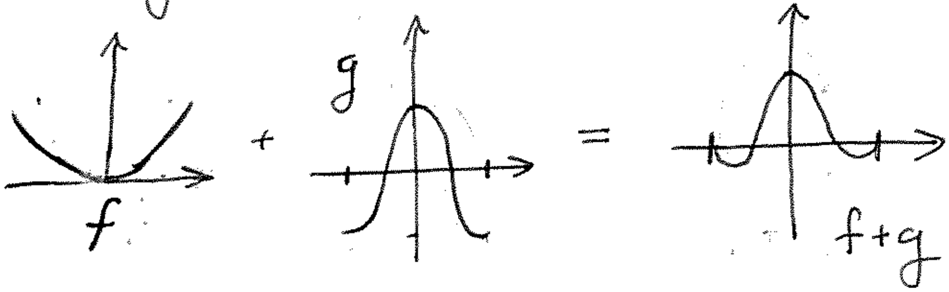
b **Infinite dim'l vector spaces.**

$$\mathcal{F} = \{\text{Continuous functions from } [-1,1] \text{ to } \mathbb{R}\}$$

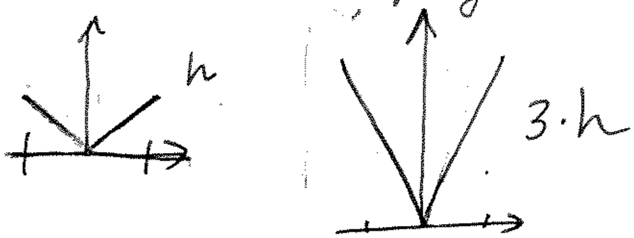
Ex: $f(x) = x^2 \quad g(x) = \cos \pi x \quad h(x) = |x|$

Note we can add them

$$(f+g)(x) = x^2 + \cos(\pi x)$$



and scalar multiply



From the right vantage point, the fact that

$$h(x) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n^2 \pi^2} \cos(\pi n x)$$

is just like that in $\mathbb{R}^3$ we have

$$\vec{v} = (3, -1, 2) = 3\vec{i} + (-1)\vec{j} + 2\vec{k}$$

where the fns $\{\cos(n\pi x)\}_{n=0}^{\infty}$ in $\mathcal{F}$ play the

the role of $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$. The fact that

$\vec{v} \cdot \vec{\imath} = 3$ is the coeff on $\vec{\imath}$ corresponds precisely

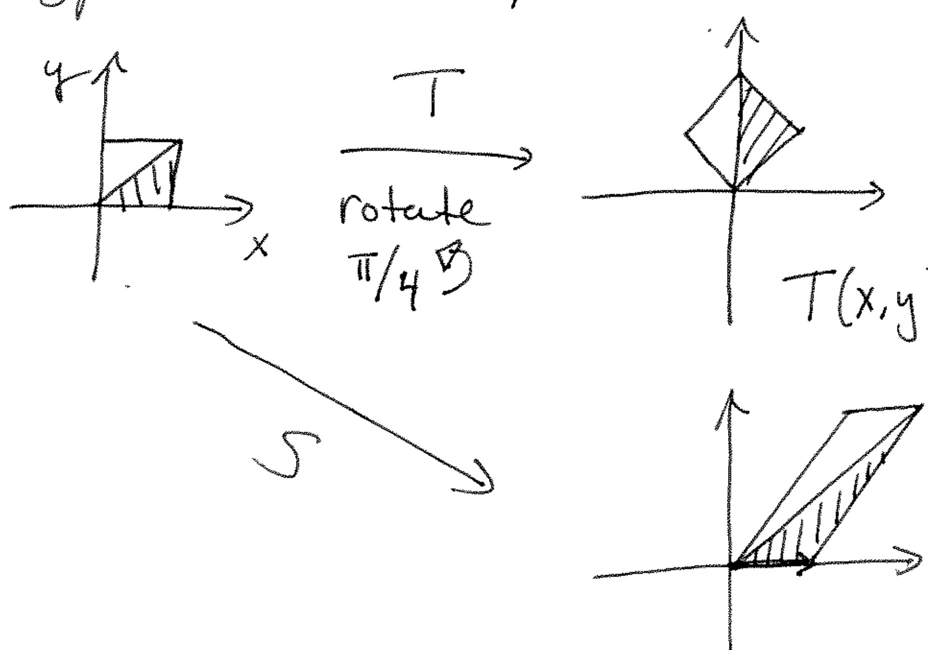
to the result

$$\frac{1}{2} \int_{-1}^{1} h(x) \cos(n \pi x) \, dx = \begin{cases} 1/2 & n = 0 \\ 0 & n \neq 0 \text{ even} \\ \dfrac{-4}{n^2 \pi^2} & n \text{ odd} \end{cases}$$

Such things are called Fourier series and in this case we're breaking the sawtooth wave into its constituent tones...

<u>Linear</u> trans<u>formations</u>: functions between vector spaces that "respect the structure"

<u>Ex</u>:



$T(x,y) = \sqrt{2}(x-y, x+y)$

$S(x,y) = (x+y, 2y)$

These are the simplist class of functions from $\mathbb{R}^a$ to $\mathbb{R}^b$. To have any hope of understanding "real world" functions, we must start here. There will be plenty to study...

To Be Continued...